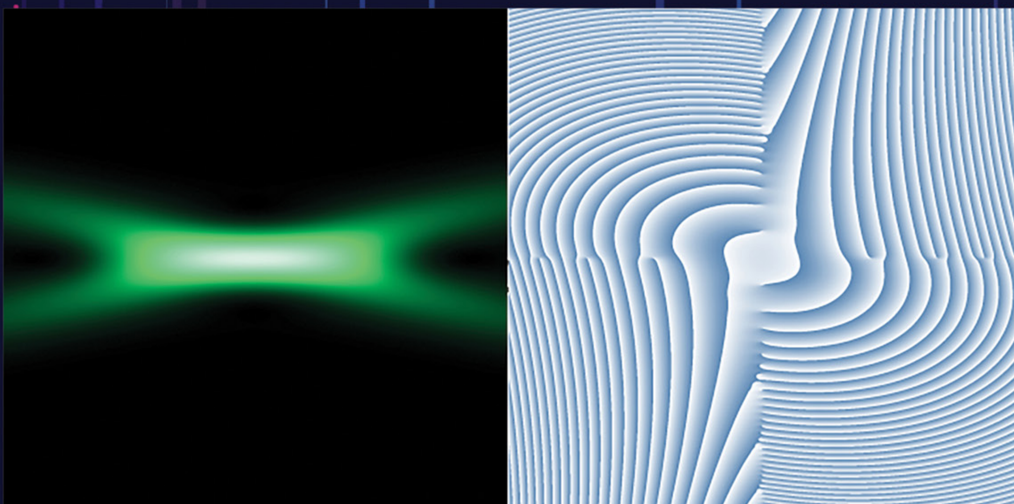
Topological Charge of Optical Vortices



Victor V. Kotlyar
Alexey A. Kovalev
Anton G. Nalimov

CRC Press
Taylor & Francis Group

Topological Charge of Optical Vortices

This book is devoted to the consideration of unusual laser beams – vortex or singular beams. It contains many numerical examples, which clearly show how the phase of optical vortices changes during propagation in free space, and that the topological charge is preserved.

Topological Charge of Optical Vortices shows that the topological charge of an optical vortex is equal to the number of screw dislocations or the number of phase singularities in the beam cross-section. A single approach is used for the entire book: based on M. Berry's formula. It is shown that phase singularities during beam propagation can be displaced to infinity at a speed greater than the speed of light. The uniqueness of the book is that the calculation of the topological charge for scalar light fields is extended to vector fields and is used to calculate the Poincare–Hopf singularity index for vector fields with inhomogeneous linear polarization with V-points and for the singularity index of vector fields with inhomogeneous elliptical polarization with C-points and C- lines.

The book is written for opticians, and graduate students interested in an interesting section of optics – singular optics. It will also be of interest to scientists and researchers who are interested in modern optics. In order to understand the content of the book, it is enough to know paraxial optics (Fourier optics) and be able to calculate integrals.

Topological Charge of Optical Vortices

Victor V. Kotlyar, Alexey A. Kovalev,
and Anton G. Nalimov

CRC Press
Taylor & Francis Group
Boca Raton London New York

CRC Press is an imprint of the
Taylor & Francis Group, an **informa** business

The authors thank Alexey Kovalev for providing the cover image.

First edition published 2023
by CRC Press
6000 Broken Sound Parkway NW, Suite 300, Boca Raton, FL 33487-2742

and by CRC Press
4 Park Square, Milton Park, Abingdon, Oxon, OX14 4RN

CRC Press is an imprint of Taylor & Francis Group, LLC

ISBN: 978-1-032-34553-6 (HB)
ISBN: 978-1-032-35309-8 (PB)
ISBN: 978-1-003-32630-4 (EB)

DOI: 10.1201/9781003326304

Typeset in Times
by Deanta Global Publishing Services, Chennai, India

Contents

Preface

The book is devoted to the consideration of unusual laser beams – vortex or singular beams. They are called vortex beams because their energy flow rotates in a spiral around the optical axis when the beam propagates along this axis. And they are called singular beams because in the cross-section of such beams there are isolated points with zero intensity, in which the phase is not defined. These points are singular (unusual points). Vortex laser beams are described by two integral (averaged) characteristics: orbital angular momentum (OAM) and topological charge (TC). The OAM of the beam shows with what moment of force the light will act on a microparticle placed at the focus of the vortex laser beam. And the TC of the beam shows how many complete jumps by 2π the phase will acquire in the cross-section of the beam when it is traversed along a closed contour that encompasses the entire beam. If the OAM of the beam is preserved during the propagation of a scalar paraxial laser beam, then the TC is also preserved if we take into account the phase singularities located at the periphery in the beam cross-section (at infinity). The OAM normalized to the beam power can be either an integer or a fractional number. And the TC of the beam (except for the initial plane and at infinity) is always an integer. In the book, the TC of the axial superposition of Laguerre–Gauss (LG) vortex beams is calculated. It turned out to be equal to the maximal TC of the beams in the superposition. Therefore, if the amplitude of the light field is expanded according to the basis of the LG beams in an infinite series, then the TC of such a light field is infinite. The book contains examples of vortex beams with infinite TC. Note that the OAM of the beam is determined by the amplitude and phase, and TC is determined only by the phase of the beam. Therefore, there are light beams in which the TC is zero, and the OAM is nonzero (for example, elliptical Gaussian beams). And vice versa, there are beams in which the OAM is equal to zero, and the TC is nonzero. The book shows that the phase singularities of optical vortices can "go" to infinity (and "come" from infinity) with a speed greater than the speed of light in a vacuum. Examples of light beams with a half-integer TC are also given, but the fractional part of the TC is at infinity ("hidden" phase). The book shows that the TC of the superposition of two parallel LG beams depends on which beam is on the left and which is on the right. And if we rearrange the LG beams in places (left to right, and right to left), then the TC of such a superposition will change to 1. The book will be of interest to everyone who is interested in optics.

A sufficient number of monographs are devoted to the study of the OAM of vortex beams, and there are no monographs on the study of TC. This book is precisely devoted to the study of the topological charge of vortex laser beams.

Acknowledgments

This book contains research results financially supported by the Russian Science Foundation grant No. 18-19-00595.

The authors are grateful to PhD A.P. Porfirev and Prof. S. Rasouli for carrying out experiments on the propagation of beams with optical vortices.

Authors

Victor V. Kotlyar is Head of the Laboratory at Image Processing Systems Institute of the Russian Academy of Science, a branch of the Federal Scientific Research Center "Crystallography and Photonics", and Professor of Computer Science at Samara National Research University, Russia. He earned his MS, PhD, and DrSc degrees in Physics and Mathematics from Samara State University (1979), Saratov State University (1988), and Moscow Central Design Institute of Unique Instrumentation, the Russian Academy of Sciences (1992). He is a SPIE- and OSA-member. He is co-author of 400 scientific papers, 7 books, and 7 inventions. His current interests are diffractive optics, gradient optics, nanophotonics, and optical vortices.

Alexey A. Kovalev graduated in 2002 from Samara National Research University, Russia, majoring in Applied Mathematics. He earned his PhD in Physics and Maths in 2012. He is senior researcher of Laser Measurements at the Image Processing Systems Institute of the Russian Academy of Science, a branch of the Federal Scientific Research Center "Crystallography and Photonics". He is a co-author of more than 270 scientific papers. His research interests are mathematical diffraction theory, photonic crystal devices, and optical vortices.

Anton G. Nalimov graduated from Samara State Aerospace University, Russia, in February 2003. He entered postgraduate study in 2003 with a focus on the specialty 05.13.18 "Mathematical Modeling and Program Complexes". He finished it in 2006 with the specialty 01.04.05 "Optics". Nalimov works in the Technical Cybernetics department at Samara National Research University as an associate professor, and also works as a scientist in the Image Processing Systems Institute of the Russian Academy of Science, a branch of the Federal Scientific Research Center "Crystallography and Photonics" in Samara. He is a PhD candidate in Physics and Mathematics, co-author of 200 papers and 3 inventions.

Introduction

Laser optical vortices are light fields that have singularity points in the phase distribution, i.e. points where the phase is undetermined. Optical vortices also have screw dislocations in their wavefront. In the intensity distribution of optical vortices, there are isolated points of zero intensity. Topological charge (TC) is one of the main characteristics of optical vortices. This is an integer number equal to the number of phase jumps by 2π along an infinite-radius circle in the beam cross-section. The TC is positive if the phase increases counterclockwise, and TC is negative otherwise. This definition of the topological charge was given by M.V. Berry and it shows that the cross-section of an optical vortex can contain both a finite and an infinite number of singularity points (local optical vortices), that can reside in the periphery of the laser beam in areas with almost zero intensity. Such peripheral points of phase singularity cannot be detected experimentally, but they contribute to the TC and cannot be neglected. Optical vortices can have only an integer TC or an indefinite TC. An optical vortex can have a fractional TC only in the initial plane, since, in this plane, arbitrary TC can be given. However, on propagation in free space, the initial fractional TC generates an infinite number of local optical vortices with opposite TCs, which are located at different distances from the optical axis of the beam. Therefore, the TC of such a beam is indefinite, since there are different numbers of phase jumps by 2π on transverse circles with different radii. The TC is conserved on propagation in free space, similarly to the orbital angular momentum of vortex beams. There are works where the authors demonstrated that the TC of the combined beams and of the beams with an initial fractional TC is not conserved on propagation. However, these works did not take into account local optical vortices located in the beam periphery, since these vortices cannot be detected experimentally or in simulation within the paraxial limits. Such peripheral optical vortices can be detected by nonparaxial simulation by using the Rayleigh–Sommerfeld integrals.

In this book, topological charges are obtained for a superposition (coaxial and noncoaxial) of the Laguerre–Gaussian and Bessel–Gaussian beams, for asymmetric beams, and the Hermite–Gaussian vortex beams. The TC evolution is shown for two combined Laguerre–Gaussian beams with different waists radii. It is shown how the TC is generated in optical vortices with an initial fractional topological charge. It is demonstrated that the TC is conserved on propagation in space, as well as after passing through an arbitrary amplitude mask, and is resistant to random phase distortions.

Taylor & Francis
Taylor & Francis Group
http://taylorandfrancis.com

1 Topological Charge of Superposition

Conservation of Topological Charge

1.1 TOPOLOGICAL CHARGE AND ASYMPTOTIC PHASE INVARIANTS OF VORTEX LASER BEAMS

There are several well-known non-diffracting and propagation-invariant light fields. The most prominent examples in 3D space are the Bessel beams [1], parabolic beams [2], Mathieu beams [3], as well as Laguerre–Gaussian and Hermite–Gaussian paraxial modes [4]. In 2D space, there are also the Airy and Weber beams [5,6]. A thorough review of propagation-invariant fields can be found in [7,8]. Besides propagating in free space, the interaction of such beams with matter is also studied, including non-linear processes [9]. Potential applications of such beams are, for example, wireless communications and optical interconnections. In addition to the beams that preserve their shape on propagation, there are several properties of the beam cross-section which are also propagation-invariant. These properties can be used as indicators that can help the receiver to identify the incoming signal beam. For instance, well-known indicators are the orbital angular momentum (OAM) and the topological charge (*TC*) of vortex beams. Many works were dedicated to the conservation of these properties, either on propagation in atmospheric turbulence [10] or after amplitude distortions [11]. Many of these works were about determining the OAM [12,13] or *TC* [14,15] of an optical signal beam.

These two indicators are often used interchangeably since for conventional rotationally symmetric optical vortices both OAM and *TC* give the same value. However, the nature of these indicators is quite different physically. While OAM is an integral property of a light field transverse intensity and phase distributions which are calculated by integration over the whole transverse plane [16,17,18], *TC* is a purely phase property which is calculated by integration of phase angular derivative over an infinite-radius circle [19] or closed curve. Thus, *TC* can be treated as an asymptotic phase property. Propagation invariance of the OAM can be easily proven mathematically since the propagation operator (Fresnel transform) is unitary. Conservation of both the OAM and of the spin angular momentum on free-space propagation was proven in [20] (Section 4 "Eigenoperator description of laser beams"). The invariance of the *TC* cannot be proven in this way. This can be proven intuitively since it

DOI: 10.1201/9781003326304-1

is known that phase singularities can disappear only as the result of the annihilation of two singularities of opposite topological charge [21]. Thus, on propagation, *TC* should not change its value. However, there is a well-known work by M.S. Soskin et al., where a superposition of Gaussian and Laguerre–Gaussian beams with different-waist radii can change the *TC* [22]. This seems to contradict the idea of *TC* conservation. Recently, we revisited this problem, studying the *TC* change on the propagation of two different-waist LG beams [23] and showed that, when nearing the *TC*-change plane, certain vortices move away from the optical axis to infinity. Thus, the summary *TC* of all the vortices, including those in infinity, remains. Therefore, two questions arise: can the *TC* conservation be proven mathematically, and are there some other propagation-invariant asymptotic phase invariants of light fields?

In this section, to prove that *TC* is conserved upon propagation, we introduce a huge-ring approximation, which is similar to the paraxial approximation, but, on the contrary, the distance from the optical axis is much larger than the propagation distance. Using this approximation, we prove that *TC* value does not change from one transverse plane to another. In addition, we show that another asymptotic phase propagation-invariants can be constructed similarly to the *TC*.

1.1.1 Orbital Angular Momentum and Topological Charge

If a light field propagates along the optical axis z and has the complex amplitude $E(r, \varphi, z)$, where (r, φ, z) are the cylindrical coordinates, then its normalized OAM (OAM J_z divided by beam power W) in a transverse plane reads as [16,17,18]:

$$\frac{J_z}{W} = \frac{\operatorname{Im}\int_0^\infty\int_0^{2\pi} E^*(r,\varphi,z)\dfrac{\partial E(r,\varphi,z)}{\partial\varphi} r dr d\varphi}{\int_0^\infty\int_0^{2\pi} E^*(r,\varphi,z)E(r,\varphi,z) r dr d\varphi}, \tag{1.1}$$

with Im being the imaginary part of a complex number, while *TC* μ is defined as the integral over an infinite-radius circle [19]:

$$\mu = \frac{1}{2\pi}\lim_{r\to\infty}\int_0^{2\pi} \frac{\partial}{\partial\varphi}\left[\arg E(r,\varphi,z)\right] d\varphi. \tag{1.2}$$

1.1.2 Propagation of a Light Field in Free Space and Conservation of Its Orbital Angular Momentum

The complex amplitude of a monochromatic light field in homogeneous medium obeys the Helmholtz equation, which in the cylindrical coordinates reads as:

$$\frac{\partial^2 E}{\partial r^2} + \frac{1}{r}\frac{\partial E}{\partial r} + \frac{1}{r^2}\frac{\partial^2 E}{\partial\varphi^2} + \frac{\partial^2 E}{\partial z^2} + k^2 E = 0, \tag{1.3}$$

where $k = 2\pi/\lambda$ is the module of wavevector for light with the wavelength of λ. For paraxial propagation, the Helmholtz equation reduces to:

$$2ik\frac{\partial E}{\partial z}+\frac{\partial^2 E}{\partial r^2}+\frac{1}{r}\frac{\partial E}{\partial r}+\frac{1}{r^2}\frac{\partial^2 E}{\partial \varphi^2}=0. \tag{1.4}$$

It is well-known that if E is a solution of Equation (1.4) then the complex amplitude $E(r, \varphi, z)$ in a transverse plane is related to that in the initial plane ($z = 0$) by the Fresnel transform [24]:

$$\begin{aligned}E(r,\varphi,z)&=\frac{-ik}{2\pi z}\exp\left(ikz+\frac{ikr^2}{2z}\right)\\&\times\int_0^\infty\int_0^{2\pi}E(\rho,\theta,0)\exp\left[\frac{ik\rho^2}{2z}-i\frac{k}{z}r\rho\cos(\theta-\varphi)\right]\rho d\rho d\theta.\end{aligned} \tag{1.5}$$

It can be shown the dot product of two functions is equal to the dot product of their Fresnel transforms. In addition, if E is a solution of Equation (1.4) then it is obvious that the functions E^* and $\partial E/\partial \varphi$ are also solutions of Equation (1.4). Therefore, both numerator and denominator in Equation (1.1) are conserved on propagation and thus the normalized OAM is conserved too. The detained proof of the OAM conservation can be found in [20] (Section 4 "Eigenoperator description of laser beams").

As to nonparaxial free-space propagation, the OAM should be analyzed as a vectorial quantity. Its z-component reads as [25]:

$$J_z=\operatorname{Im}\sum_{i=x,y,z}\int_0^\infty\int_0^{2\pi}E_i^*(r,\varphi,z)\frac{\partial E_i(r,\varphi,z)}{\partial\varphi}rdrd\varphi, \tag{1.6}$$

$$E(x,y,z)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}A(\alpha,\beta)\exp\left[ik\left(\alpha x+\beta y+z\sqrt{1-\alpha^2-\beta^2}\right)\right]d\alpha d\beta, \tag{1.7}$$

with (x, y) being the Cartesian coordinates in a transverse plane and (α, β) being the Cartesian coordinates in the Fourier plane (cosines of the angles defining the directions of plane waves), $A(\alpha, \beta)$ being the angular spectrum of plane waves.

1.1.3 Conservation of the Topological Charge

Unfortunately, the conservation of TC (Equation (1.2)) cannot be proven so easily. According to the TC definition in Equation (1.2), the field should be analyzed in its periphery, at an infinite distance r from the optical axis. Thus, a paraxial approximation is inappropriate here. Therefore, we introduce another approximation here, quite opposite to the paraxial. Generally, without the paraxial limits, if a light field propagates along the z-axis, its complex amplitudes in two transverse planes (source plane and observation plane) are related by the Rayleigh–Sommerfeld integral [26]:

$$E(r,\varphi,z) = \frac{-1}{2\pi}\int_0^{\infty}\int_0^{2\pi} E(\rho,\theta,0)\frac{\partial}{\partial z}\left[\frac{\exp(ikL)}{L}\right]\rho d\rho d\theta, \tag{1.8}$$

where L is the distance between a point in the source plane (ρ , θ , 0) and a point in the observation plane (r, φ , z):

$$L = \left[z^2 + r^2 + \rho^2 - 2r\rho\cos(\theta-\varphi)\right]^{1/2}. \tag{1.9}$$

The complex amplitude given by Equation (1.8) is an exact solution of the Helmholtz equation and describes a light field without paraxial approximation. The Fresnel transform in Equation (1.5) can be obtained from Equation (1.8) for a case when the propagation distance is large compared to the transverse coordinates (paraxial propagation) (Figure 1.1(a)) and therefore:

$$L \approx z + \frac{r^2 + \rho^2 - 2r\rho\cos(\theta-\varphi)}{2z}. \tag{1.10}$$

To calculate the topological charge, we now suppose that, on the contrary, r is much greater than z and ρ (Figure 1.1(b)). Thus, the distance L is given by:

$$L \approx r + \frac{z^2 + \rho^2}{2r} - \rho\cos(\theta-\varphi). \tag{1.11}$$

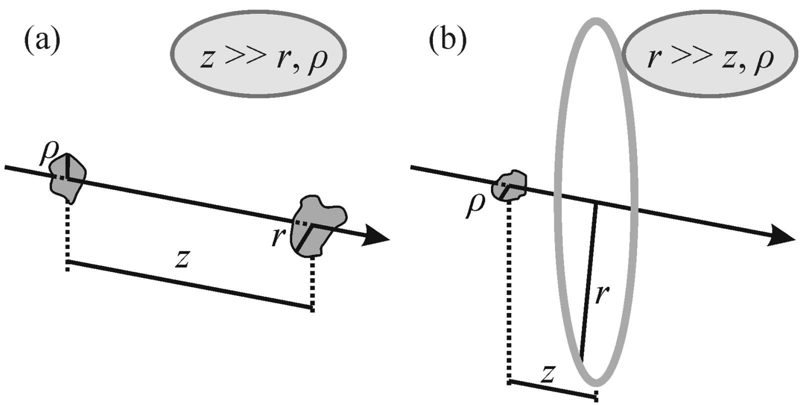


FIGURE 1.1 (a) Paraxial approximation: propagation distance z is much larger than transverse coordinates ρ and r in the input and output planes respectively, (b) huge-ring approximation: transverse coordinate r in the output plane is much larger than the propagation distance z and the transverse coordinate ρ in the input plane.

We call here this expression a huge-ring approximation. Using it, we obtain the complex amplitude on a very large radius ring in a transverse plane. The Rayleigh–Sommerfeld integral can be rewritten as:

$$E(r,\varphi,z)=\frac{-z}{2\pi}\int_0^{\infty}\int_0^{2\pi}E(\rho,\theta,0)\left(\frac{ik}{L^2}-\frac{1}{L^3}\right)\exp(ikL)\,\rho d\rho d\theta. \tag{1.12}$$

Similarly to the derivation of the Fresnel transform, we use Equation (1.11) for the exponent in the integrand, while the expression ($ik/L^2 - 1/L^3$) we write approximately as ik/r^2:

$$\begin{aligned}E(r,\varphi,z)&=\frac{-ikz}{2\pi r^2}\exp\left(ikr+ik\frac{z^2}{2r}\right)\\&\times\int_0^{\infty}\int_0^{2\pi}E(\rho,\theta,0)\exp\left[\frac{ik\rho^2}{2r}-ik\rho\cos(\theta-\varphi)\right]\rho d\rho d\theta.\end{aligned} \tag{1.13}$$

Equation (1.13) is the main equation in this section and it allows us to estimate the complex amplitude on a circle of a very large radius, just as needed to calculate *TC*. It is seen in Equation (1.13) that the integral is independent of z and depends only on the angular polar coordinate φ . On the contrary, the multipliers before the integrals are φ -independent. Therefore, the φ -derivative of the field phase $\partial(\arg E)/\partial\varphi$ is independent of z:

$$\frac{\partial}{\partial z}\left[\lim_{r\to\infty}\frac{\partial}{\partial\varphi}\arg E(r,\varphi,z)\right]=0. \tag{1.14}$$

Consequently, z-independent are any quantities obtained from $\partial(\arg E)/\partial\varphi$ at large radii r. The most prominent example is the topological charge of Equation (1.2). Thus, *TC* conservation is just a partial case following Equation (1.14), and below we consider some other partial cases.

1.1.4 Asymptotic Phase Invariants of Vortex Laser Beams

Since the field should be continuous at $\varphi=0$ and $\varphi=2\pi$, $\arg E$ should change by an integer number of 2π , thus forcing *TC* to be an integer number. Even if in the initial plane it was fractional, then, on propagation, it becomes an integer [19]. However, Equation (1.14) indicates that, by using the function $\partial(\arg E)/\partial\varphi$ at large radii r, other propagation-invariant quantities may also be constructed, e.g.:

$$\mu_g=\frac{1}{2\pi}\lim_{r\to\infty}\int_0^{2\pi}g(\varphi)\frac{\partial}{\partial\varphi}\left[\arg E(r,\varphi,z)\right]d\varphi \tag{1.15}$$

with $g(\varphi)$ being an arbitrary function. To confirm our theory, we made some numerical experiments. For constructing the invariants, we choose two functions: $g_1(\varphi) = \cos\varphi$ and $g_2(\varphi) = \mathrm{rect}(\varphi/\pi)$ (i.e. $g_2(\varphi) = 1$ at $-\pi/2 \leq \varphi \leq \pi/2$ and $g_2(\varphi) = 0$ otherwise). So, we test whether or not the following values:

$$\mu_1 = \frac{1}{2\pi}\lim_{r\to\infty}\int_0^{2\pi}\cos\varphi\frac{\partial}{\partial\varphi}\left[\arg E(r,\varphi,z)\right]d\varphi, \tag{1.16}$$

$$\mu_2 = \frac{1}{2\pi}\lim_{r\to\infty}\int_{-\pi/2}^{\pi/2}\frac{\partial}{\partial\varphi}\left[\arg E(r,\varphi,z)\right]d\varphi \tag{1.17}$$

are asymptotic phase invariants. We note that, in general, the functions $g(\varphi)$ and the argument of the complex amplitude (arg E) in Equation (1.15) are not analytic functions (for instance, $g_2(\varphi)$ in Equation (1.17) is not analytic). Therefore, the integral within Equation (1.15) cannot be evaluated by transforming to a contour integral in the complex plane and by using the theory of residues.

1.1.5 Numerical Simulation

The theory is obtained for the invariant quantities, computed over an infinite-radius circle. This is impractical, but gives an idea that these quantities can also conserve in other, realistic, conditions. Below, we consider two paraxial light beams and choose feasible simulation parameters quite opposite to the huge-ring approximation, i.e. when the circle radius is much smaller than the propagation distance. For a test beam, we take a superposition of two Gaussian beams with optical vortices. In the initial plane ($z=0$), such a beam has the following complex amplitude:

$$E(r,\varphi,0) = \exp\left(-\frac{r^2}{w^2}\right)\left[A_1\exp(in_1\varphi) + A_2\exp(in_2\varphi)\right], \tag{1.18}$$

where w is the Gaussian beam waist radius, n_1 and n_2 are the topological charges of the vortices, A_1 and A_2 are the superposition coefficients.

Figure 1.2 illustrates the intensity and phase distributions of the test beam (Equation 1.18) in several transverse planes for the following parameters: wavelength $\lambda = 633$ nm, waist radius $w = 1$ mm, topological charges $n_1 = 10$ and $n_2 = 5$, superposition coefficients $A_1 = 1$ and $A_2 = 0.5$, propagation distances $z = 0$ (initial plane), $z = z_0/2$ ($z_0 = kw^2/2$ is the Rayleigh range), and $z = 2z_0$ (far field). The calculation area is $-R \leq x, y \leq R$ with (x, y) being the Cartesian coordinates and R being the half-size of the area: $R = 5$ mm for $z = 0$, $R = 10$ mm for $z = z_0/2$, and $R = 20$ mm for $z = 2z_0$. Dashed lines on the phase distributions show the circles along which we calculate TC (Equation (1.2)) and the invariants shown in Equations (1.16) and (1.17) (all circles are of radius $0.8R$). Figure 1.2(a, d) is obtained by Equation (1.18), while Figure 1.2(b, c, e, f) is obtained by using an expression for the Fresnel diffraction of a Gaussian

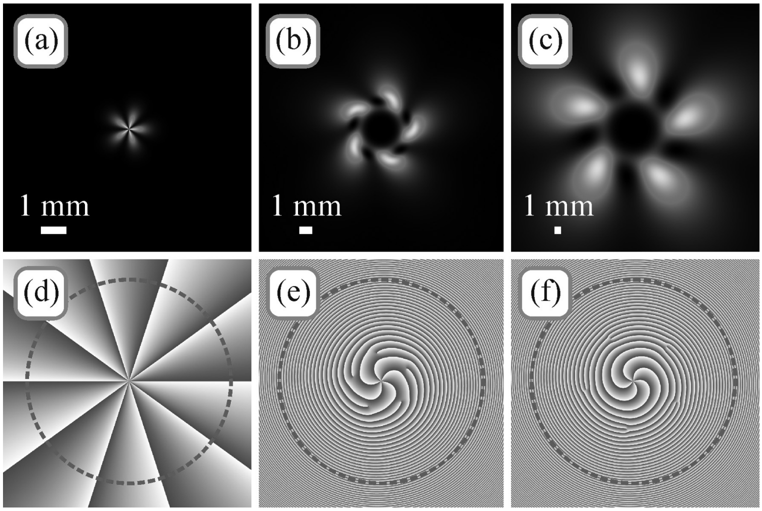


FIGURE 1.2 Intensity (a–c) and phase (d–f) distributions of a superposition of two Gaussian vortices (Equation (1.18)) in several transverse planes for the following parameters: wavelength $\lambda = 633$ nm, waist radius $w = 1$ mm, topological charges $n_1 = 10$ and $n_2 = 5$, superposition coefficients $A_1 = 1$ and $A_2 = 0.5$, propagation distances $z = 0$ (a, d), $z = z0/2$ (b,e), and $z = 2z0$ (c,f), calculation area $-R \le x, y \le R$ with R = 5 mm (a, d), R = 10 mm (b, e), and R = 20 mm (c, f).Dashed circles on the phase distributions are those along which *TC* (Equation (1.2)) and invariants from Equations (1.16) and (1.17) are calculated.

optical vortex [27,28]. Calculation of *TC* by Equation (1.2) yields nearly the same values for all three propagation distances: $\mu = 9.9999$ at $z = 0$, $\mu = 9.9695$ at $z = z_0/2$, and at $z = 2z_0$ (theoretical value of μ is 10 [13]). Calculation by Equation (1.16) yields the value $\mu_1 = -0.0015$ for all three distances z. Calculation by Equation (1.17) yields the value $\mu_2 = 4.8514$ at $z = 0$, $\mu_2 = 4.9702$ at $z = z_0/2$, and $\mu_2 = 4.9654$ at $z = 2z_0$ (it is obvious that the theoretical value of μ_2 should be 10/2 = 5). As another test beam, we consider a Gaussian beam with multiple vortices located on a circle of a radius r_0. Such a beam can be obtained as a coaxial superposition of a Laguerre–Gaussian vortex beam and of a Gaussian beam [29]. The complex amplitude of this beam reads as:

$$E(r,\varphi,z) = \frac{1}{q}\exp\left(-\frac{r^2}{qw^2}\right)\left(\frac{r^m e^{im\varphi}}{q^m} - r_0^m\right), \tag{1.19}$$

where w is the Gaussian beam waist radius, m is the number of optical vortices with unit topological charge, $q = 1 + iz/z_0$.

Figure 1.3 depicts the intensity and phase distributions of the test beam from Equation (1.19) in several transverse planes for the following parameters: wavelength $\lambda = 633$ nm, waist radius $w = 1$ mm, number of vortices $m = 3$, the radius of the circle with vortices $r_0 = 0.7w$, propagation distances $z = 0$, $z = z_0/2$, and $z = 2z_0$, calculation area $-R \le x, y \le R$ with $R = 5$ mm for $z = 0$, $R = 10$ mm for $z = z_0/2$, and $R = 20$ mm

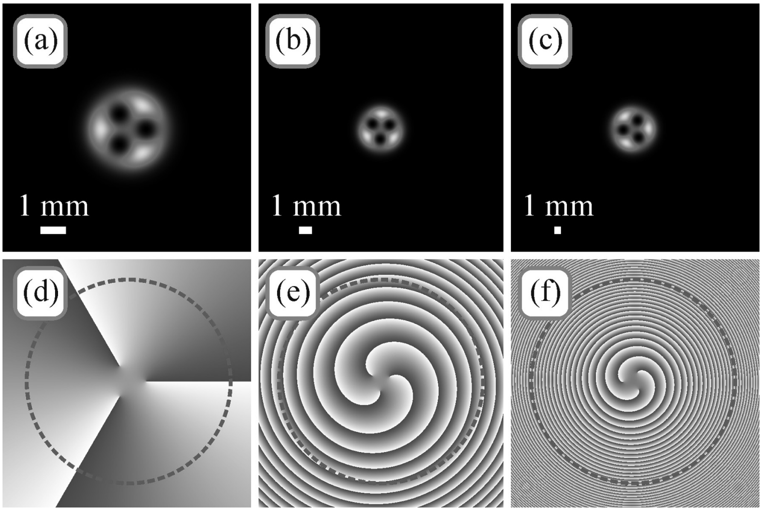


FIGURE 1.3 Intensity (a–c) and phase (d–f) distributions of a Gaussian beam with several vortices (Equation (1.19)) in several transverse planes for the following parameters: wavelength $\lambda = 633$ nm, waist radius $w = 1$ mm, number of vortices $m = 3$, radius of the circle with vortices $r_0 = 0.7w$, propagation distances $z = 0$ (a,d), $z = z_0/2$ (b, e), and $z = 2z_0$ (c, f), calculation area $-R \leq x, y \leq R$ with R = 5 mm (a, d), R = 10 mm (b, e), and R = 20 mm (c, f). Dashed circles on the phase distributions are those along which *TC* (Equation (1.2)) and invariants from Equations (1.16) and (1.17) are calculated.

for $z = 2z_0$. Dashed lines on the phase distributions show the circles along which we calculate *TC* (Equation (1.2)) and the invariants in Equations (1.16) and (1.17) (all circles are of radius $0.8R$). All patterns in Figure 1.3 are obtained by Equation (1.19).

Calculation of *TC* by Equation (1.2) yields $\mu = 3.0000$ at $z = 0$, $\mu = 2.9996$ at $z = z_0/2$, and $\mu = 2.9941$ at $z = 2z_0$ (theoretical value of μ is 3). Calculation by Equation (1.16) yields the value $\mu_1 \approx 0$ ($\mu_1 \sim 10^{-4}$) for all three distances z. Calculation by Eq. (1.17) yields the value $\mu_2 = 1.4980$ at $z = 0$, $\mu_2 = 1.4995$ at $z = z_0/2$, and $\mu_2 = 1.4970$ at $z = 2z_0$ (i.e. $\mu_2 \approx 1.5$ in all transverse planes).

Thus, we note that despite the above theory proving conservation of the *TC* and other asymptotic phase invariants, when they are calculated over an infinite-radius circle and when the light field propagates by a finite distance, the simulation; however, demonstrates that, in practice, for some specific light fields these quantities can be conserved even when calculated over circles comparable to the beam transverse sizes, and much smaller than the propagation distance. This gives a potential for using these quantities for identifying incoming signals in optical wireless communications. It is hardly possible to estimate the necessary circle radius (R relative to z) for an arbitrary beam, but in all parts of Figures 1.2 and 1.3, this radius is about several times the effective beam width.

In conclusion, we have suggested in this section an alternative way to prove the conservation of the topological charge of a light field on propagation [30]. Our proof is based on a so-called huge-ring approximation of the Huygens–Fresnel principle, which is opposite to the paraxial approximation and which we suggested here for the observation point on an infinite-radius ring. It turned out, that, in addition to the topological charge, phase distribution in areas far from the optical axis allows obtaining other quantities that are also propagation-invariant and the number of these invariants is theoretically infinite. In a simulation, we suggested two such invariants and tested them on two paraxial light fields. Of course, of practical interest are the invariants that conserve the rings of finite radii, and in the simulation we also used finite-radius rings. However, there may exist light fields, for which these invariants fail to conserve. Thus, construction of the propagation-invariants by using the condition (Equation 1.15), investigating their applicability to various-kind light fields, and developing the methods for measuring these invariants, is yet to be studied. The results of this work can find applications in optical data transmission. In general cases, for correct measuring of the *TC* of non-symmetric incoming optical signal, phase distribution should be obtained, for example, by the Shack–Hartmann wavefront sensor, as was studied experimentally in [31,32]. Identifying an incoming beam by using the invariants, like the partial *TC* (Equation 1.17), allows measuring the wavefront in a smaller area (rather than over the whole circle in the beam periphery).

1.2 TOPOLOGICAL CHARGE OF A LINEAR COMBINATION OF OPTICAL VORTICES: TOPOLOGICAL COMPETITION

Laser optical vortices (OV) are a particular type of laser beam that carries an orbital angular momentum (OAM) [33]. The OAM associated with paraxial, nonparaxial, and vector beams has been amply studied, as can be seen from works published in 2019 alone [34,35,36,37,13,38,39]. Well-known examples of laser OV are presented by Laguerre–Gauss modes [4], Bessel [1], Bessel–Gauss beams [40], Hypergeometric [41], and Circular [42] beams. The listed radially symmetric beams carry the same OAM normalized to the beam power, which is equal to the beams' integer *TC*, n. Non-axially symmetric OVs have also been described and known to carry different OAM, for which a variety of formulae have been deduced [43,44]. As well as carrying OAM, optical vortices are also characterized by the topological charge *TC*, which was defined in [19]. The literature dealing with the calculation of *TC* of composite OVs is very scarce. For example, in [10], *TC* was shown to conserve in a medium-turbulence atmosphere over a distance of several kilometers, while in [45], *TC* variations were numerically studied of vortex soliton in a non-linear medium. Sometimes authors identify *TC* and OAM, moreover, sometimes they assert that *TC* can be changed via diffraction by simple dielectric obstacles. On the other hand, in the articles [46,11] it was shown that the sectorial aperture can significantly change the OAM while the *TC* remains constant and equal to the initial value. Thus, it became necessary to provide some clarification on this issue.

In this section, we focused on some noteworthy examples of the *TC* behavior in optical vortex arrays that show a cautious approach to calculating *TC*. In particular, we demonstrate that *TC* of an optical vortex is conserved in spite of amplitude distortion and shift of the OV center across the carrier beam. It is also shown that in a linear superposition of simple OVs whose amplitude is given by $A(r)\exp(in\varphi)$ (where (r, φ) are the polar coordinates in the beam cross-section) the constituent beams enter a "competition": *TC* of the resulting beam is defined by both magnitude and sign of the constituent vortex, $+n$, $-n$, as well as being dependent on the amplitude of weight coefficients of the linear combination.

1.2.1 TC of an OV after Passing an Amplitude Mask

Below, we analyze changes in *TC* resulting from "cut-off" of a sector-shaped portion from an optical vortex. OVs with a "cut-off" sector have been discussed in detail by Volyar et al. [46]. This work has given an impetus to study the topic of *TC* conservation following different types of distortions and transformation of an OV. The definition of *TC* of an OV (and an arbitrary paraxial light field) was given by Berry [19] and J. Nye [21]. For an arbitrary light field with complex amplitude $E(r, \varphi)$, where (r, φ) are the polar coordinates, can be written in the form [19]:

$$TC = \frac{\lim}{r \to \infty} \frac{1}{2\pi} \int_0^{2\pi} d\varphi \frac{\partial}{\partial \varphi} \arg E(r,\varphi) = \frac{1}{2\pi} \frac{\lim}{r \to \infty} \operatorname{Im} \int_0^{2\pi} d\varphi \frac{\partial E(r,\varphi)/\partial \varphi}{E(r,\varphi)}. \tag{1.20}$$

This means that the *TC* monochromatic beam is specified as the total number of optical vortices in the transverse cross-section of a light stream taking into account their signs. Let us write the complex amplitude $E_n(r, \varphi)$ with a cut-off sector as:

$$E(r,\varphi) = A(r)\exp(in\varphi) f(\varphi), \tag{1.21}$$

where the sector function reads as:

$$f(\varphi) = \begin{cases} 1, -\alpha < \varphi < \alpha, \\ \delta \ll 1, \text{otherwise.} \end{cases} \tag{1.22}$$

Substituting Equation (1.21) into Equation (1.20) yields:

$$\begin{aligned} TC &= \frac{\lim}{r \to \infty} \frac{\operatorname{Im}}{2\pi} \int_0^{2\pi} d\varphi \frac{inE(r,\varphi) + A(r)e^{in\varphi} \dfrac{\partial f(\varphi)}{\partial \varphi}}{E(r,\varphi)} \\ &= \frac{\operatorname{Im}}{2\pi} \frac{\lim}{r \to \infty} \int_0^{2\pi} d\varphi \left(in + \frac{\partial f(\varphi)}{\partial \varphi} \frac{1}{f(\varphi)} \right) = n. \end{aligned} \tag{1.23}$$

The final equality in Equation (1.23) reflects the fact that the second term within the brackets is real. We may infer that if the aperture is only φ -angle dependent, *TC* of an OV remains unchanged. Although, if strictly $\delta = 0$ in Equation (1.22), then instead of Equation (1.23) it should be written that $TC = \alpha n/\pi$. But fractional TC can only be in the initial plane. The proof of Equation (1.23) can easily be repeated for an arbitrarily-shaped amplitude filter (Equation 1.22), which is defined by both angle φ and radius r:

$$f(r,\varphi) = \begin{cases} 1, (r,\varphi) \in \Omega, \\ \delta \ll 1, \ (r,\varphi) \notin \Omega, \end{cases} \tag{1.24}$$

where Ω is the diaphragm cut-off area. Then, instead of Equation (1.23), we obtain a similar relation:

$$TC = \frac{\text{Im}}{2\pi} \frac{\lim}{r \to \infty} \int_0^{2\pi} d\varphi \left(in + \frac{\partial f(r,\varphi)}{\partial \varphi} \frac{1}{f(r,\varphi)} \right) = n. \tag{1.25}$$

A weak transmission ($\delta << 1$) was introduced in Equations (1.22) and (1.24) in the region where the diaphragm should not transmit light in order to avoid the 0/0 uncertainty in the division of $E(r, \varphi)$ in Equations (1.23) and (1.25) on $E(r, \varphi)$. We note that the derivative $\partial (n\varphi)/\partial \varphi$ is equal to n only if the angle φ can be arbitrary ($0 < \varphi < 2\pi$). This means that instead of conditions in Equations (1.22) and (1.24) it is sufficient that a closed curve existed around the singular point (OV center) with nonzero field amplitude on this curve. A simulation (Figure 1.4) confirms this requirement. An indirect confirmation of the conservation of TC of an optical vortex with a cut-off

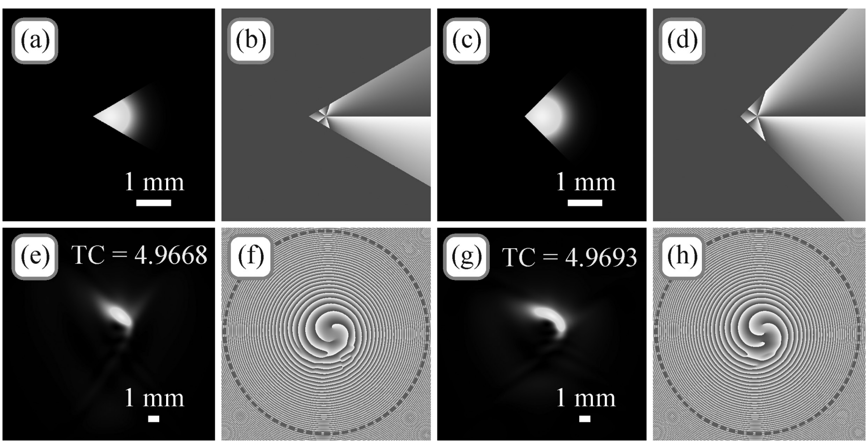


FIGURE 1.4 Distributions of intensity (a,c,e,g) and phase (b, d, f, h) of a Gaussian optical vortex bounded by a sector-shape diaphragm in the initial plane $z = 0$ (a–d) and after propagation in free space (e–h) for two different angles of the sector aperture $\alpha = \pi/6$ (a, b, e, f) and $\alpha = \pi/4$ (c, d, g, h). Dashed rings (f, h) show the circle over which the TC was calculated. White text (e, g) shows the TC.

sector Equations (1.23) and (1.25) is that the OAM of such a beam is equal to the topological charge. Indeed, the OAM normalized to the beam power:

$$J_z = \mathrm{Im}\frac{1}{2\pi}\int_0^{\infty}\int_0^{2\pi}\bar{E}(r,\varphi,z)\left(\frac{\partial E(r,\varphi,z)}{d\varphi}\right)rdrd\varphi$$

$$= \mathrm{Im}\frac{1}{2\pi}\int_0^{\infty}\int_{-\alpha}^{\alpha}A(r)e^{in\varphi}\left(in\bar{A}(r)e^{-in\varphi}\right)rdrd\varphi = \frac{\alpha n}{\pi}\int_0^{\infty}|A(r)|^2\,rdr, \tag{1.26}$$

$$\frac{J_z}{W} = n, \quad W = \frac{\alpha}{\pi}\int_0^{\infty}|A(r)|^2\,rdr.$$

It is seen from Equation (1.25) that multiplying the complex amplitude (Equation 1.21) of the OV by any real function does not change *TC* of the original OV, as a real function does not change the complex amplitude argument in Equation (1.20). Optically speaking, the multiplication of the light field amplitude by a real function is equivalent to the passage of light through a thin amplitude mask (with the above conditions taken into account).

When the spiral phase plate is bounded by a sector aperture (Equation 1.21), a Gaussian beam after passing has the following complex amplitude in the Cartesian coordinates:

$$E(x,y,0) = \exp\left[-\frac{x^2+y^2}{w^2} + in\arg(x+iy)\right]\mathrm{rect}\left\{\frac{\arg\left[(x-x_0)+i(y-y_0)\right]}{2\alpha}\right\}, \tag{1.27}$$

where w is the Gaussian beam waist radius, n is the *TC* of the spiral phase plate, α from Equation (1.22) is the half-angle of the sector aperture, (x_0, y_0) is the shift vector of the sector aperture (the singular point should be inside the sector), $\mathrm{rect}(x) = 1$ at $|x| \le 1/2$ and $\mathrm{rect}(x) = 0$ at $|x| > 1/2$, and $\arg(x)$ is meant as the principal value of the argument (i.e. $-\pi < \arg(x) \le \pi$).

Figure 1.4 shows distributions of intensity (a,c,e,g) and phase (b,d,f,h) of a Gaussian optical vortex bounded by a sector-shape diaphragm in the initial plane $z = 0$ (a–d) and after propagation in free space (e–h) for two different angles of the sector aperture $\alpha = \pi/6$ (a,b,e,f) and $\alpha = \pi/4$ (c,d,g,h). Distributions in the initial plane are obtained by Equation (1.27), while distributions at a distance are obtained by the Fresnel transform implemented numerically in MATLAB® as a convolution using the fast Fourier transform. The following parameters are used in the calculations: wavelength $\lambda = 532$ nm, Gaussian beam waist radius $w = 1$ mm, *TC* of the spiral phase plate $n = 5$, vector of the sector diaphragm shift $(x_0, y_0) = (-0.5, 0)$ mm, propagation distance $z = z_0/2$ ($z_0 = kw^2/2$ is the Rayleigh range), calculation area $-R \le x, y \le R$ ($R = 12.5$ mm, although Figure 1.4 shows smaller areas), number of pixels $N = 4096$. The obtained values are 4.9668 for $\alpha = \pi/6$ and 4.9693 for $\alpha = \pi/4$.

1.2.2 TC of an Off-Axis Optical Vortex

In this subsection, we analyze how *TC* changes upon an off-axis shift of the OV center from the optical axis of a radially symmetric beam with amplitude $A(r)$. Can an OV be shifted by an arbitrary vector (r_0, φ_0). Then, instead of Equation (1.21), the complex amplitude $E_n(r, \varphi)$ takes the form:

$$E_n\left(r,\varphi\right)=\left(\frac{re^{i\varphi}-r_0e^{i\varphi_0}}{w}\right)^n A\left(r\right). \tag{1.28}$$

Substituting Equation (1.28) into Equation (1.20) yields:

$$TC=\frac{\lim}{r\to\infty}\frac{\mathrm{Im}}{2\pi}\int_0^{2\pi}d\varphi\frac{inre^{i\varphi}}{re^{i\varphi}-r_0e^{i\varphi_0}}=\frac{1}{2\pi}\mathrm{Im}\frac{\lim}{r\to\infty}\int_0^{2\pi}d\varphi\frac{inre^{i\varphi}}{re^{i\varphi}-r_0e^{i\varphi_0}}=n. \tag{1.29}$$

The final equality in Equation (1.29) stems from the fact that for large radii ($r >> r_0$), only the first term is retained in the denominator. It is seen from Equation (1.29) that an off-axis shift of the OV center relative to a radially symmetric beam (e.g. a Gaussian beam) does not lead to a change in *TC*. In the meantime, for a beam with an off-axis phase singularity center, the normalized OAM is lower than *TC* of the whole beam, with the former decreasing with increasing shift magnitude r_0 [47,48].

Figure 1.5 shows the distribution of the intensity and phase of a Gaussian beam with an off-axis optical vortex in the initial plane and after propagation in space for different displacements of the vortex from the optical axis. The complex amplitude in the initial plane is $E_n\left(x,y\right)=\left[\left(re^{i\varphi}-r_0e^{i\varphi_0}\right)/w\right]^n\exp\left[-\left(x^2+y^2\right)/w^2\right]$, where w is the waist radius of the Gaussian beam, n and (r_0, φ_0) are the topological charge of the optical vortex, and the vector (in polar coordinates) of its displacement from the optical axis. The complex amplitude after propagation in space is calculated using a numerical Fresnel transform realized in the form of convolution using the fast Fourier transform. The following calculation parameters were used: $w=1$ mm, $n=7$, $\varphi_0=0$, $r_0=w_0/4$ (Figure 1.5(a, b)), $r_0=w_0/2$ (Figure 1.5(c, d)), $r_0=2w_0$ (Figure 1.5(e, f)), space distance $z=z_0/2$, computational domain $-R \le x, y \le R$ ($R=5$ mm). The *TC* in the initial plane, calculated numerically by the formula in Equation (1.20) (along a ring of radius $0.8R$), is 6.9997 for $r_0=w_0/4$ and $r_0=w_0/2$, 6.9995 for $r_0=2w_0$, i.e. in all cases about 7. At a distance of *TC*, it turned out to be 6.9989, 6.9989, and 6.9986, respectively.

An interesting case occurs when an optical vortex is bounded by a diaphragm in the initial plane and therefore it is impossible to use the limit $r\to\infty$ like in Equation (1.29). For example, if a spiral phase plate (SPP) is bounded by a circular diaphragm with a radius R and is shifted horizontally from the optical axis by a distance x_0, a plane wave after passing such SPP acquires the following complex amplitude:

$$E(r,\varphi)=\mathrm{circ}\left(\frac{r}{R}\right)\exp\left[in\arctan\left(\frac{r\sin\varphi}{r\cos\varphi-x_0}\right)\right], \tag{1.30}$$

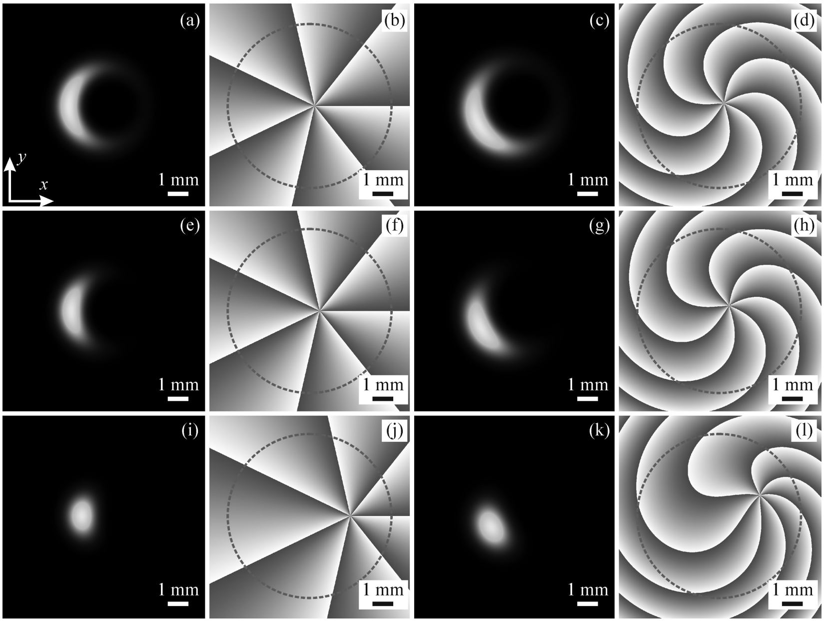


FIGURE 1.5 Distributions of intensity (a, c, e, g, i, k) and phase (b, d, f, h, j, l) of a Gaussian beam with an off-axis optical vortex in the initial plane (a, b, e, f, i, j) and after propagation in space (c, d, g, h, k, l) for different lateral displacements of the vortex from the optical axis. Calculation parameters: waist radius w = 1 mm, TC is n = 7, displacement $r_0 = w_0/4$ (a–d), $r_0 = w_0/2$ (e–h), $r_0 = 2w_0$ (i–l); $\varphi_0 = 0$ in all figures, the propagation distance in space is $z = z_0/2$ (z_0 is the Rayleigh distance). Thedashed rings on the phase distributions denote the radius of the ring by which the TC was calculated by Equation(1.20).

where circ(r/R) = 1 for $r \le R$ and circ(r/R) = 0 for $r > R$. Topological charge (Equation (1.20)) of the initial vortex field from Equation (1.30) is given by:

$$TC = \frac{n}{2\pi}\int_0^{2\pi} \frac{r^2 - rx_0\cos\varphi}{R^2 + x_0^2 - 2rx_0\cos\varphi} d\varphi = \begin{cases} n, \; x_0 < R, \\ n/2, \; x_0 = R. \end{cases} \tag{1.31}$$

Equation (1.31) illustrates that shifting the SPP center conserves TC which equals n if the SPP center is within the diaphragm.

If the SPP center is on the diaphragm edge, TC decreases two times immediately. Equation (1.31) is consistent with the condition from the previous section which states that there should not be zero amplitude around the center of singularity. Interestingly, the OAM of the beam from Equation (1.30) decreases continuously till zero when the shift distance x_0 increases from 0 to R:

$$\frac{J_z}{W} = n\left(1 - \frac{x_0^2}{R^2}\right). \tag{1.32}$$

From Equation (1.32), if $x_0 = R$, the beam's OAM is zero.

1.2.3 TC of an Optical Vortex with Multi-Center Optical Singularities

Below, we analyze a laser Gaussian beam with m embedded simple ($TC=+1$) phase singularities distributed uniformly on a circle of radius a, i.e., at points defined by the Cartesian coordinates:

$$\begin{cases} x = a\cos\varphi_p, \\ y = a\sin\varphi_p, \end{cases} \tag{1.33}$$

where $\varphi_p = 2\pi p/m$, $p = 0, \ldots, m-1$. The complex amplitude of such an OV at an arbitrary distance from the waist can be shown to be given by:

$$E(r,\varphi,z) = \frac{1}{\sigma}\left(\frac{\sqrt{2}}{w_0}\right)^m \exp\left(-\frac{r^2}{\sigma w_0^2}\right)\left(\frac{r^m e^{im\varphi}}{\sigma^m} - a^m\right), \tag{1.34}$$

where $\sigma = 1 + iz/z_0$ and $z_0 = kw_0^2/2$ is the Rayleigh range (k is the wave number). Substituting Equation (1.34) into Equation (1.20) yields:

$$TC = \frac{1}{2\pi}\lim_{r\to\infty}\mathrm{Im}\left\{\int_0^{2\pi}\frac{im\sigma^{-m}r^m e^{im\varphi}}{\sigma^{-m}r^m e^{im\varphi} - a^m}d\varphi\right\} = m. \tag{1.35}$$

Because at $r \to \infty$, the term a^m in the denominator is negligibly small, TC of the beam Equation (1.34) turns out to be independent on the distance z passed and the radius a of the circle of the OV centers, instead, being equal to the number of simple OVs in the beam. This result can be extended onto an arbitrary case of m OV centers with multiplicity m_p are found at points (r_p, φ_p), where $p = 1, 2, \ldots m$ and the carrier amplitude $A(r)$ is axially symmetric. Such a complex OV is given by the complex amplitude [49,50]:

$$E_m(r,\varphi,z=0) = A(r)\prod_{p=1}^{m}\left(re^{i\varphi} - r_p e^{i\varphi_p}\right)^{m_p}. \tag{1.36}$$

Substituting (1.36) into (1.20) yields:

$$TC = \frac{1}{2\pi}\lim_{r\to\infty}\mathrm{Im}\left\{\int_0^{2\pi} ire^{i\varphi}\sum_{p=1}^{m}\frac{m_p}{re^{i\varphi} - r_p e^{i\varphi_p}}d\varphi\right\} = \sum_{p=1}^{m} m_p. \tag{1.37}$$

Equation (1.37) suggests that in a beam with axially symmetric amplitude and several degenerate simple OVs of Equation (1.36), with their centers located at arbitrary points over the beam cross-section, TC equals the sum of multiplicity (degeneracy) values of all constituent vortices.

1.2.4 TC of an On-Axis Combination of Optical Vortices

Here, we discuss a light field whose complex amplitude is described by a linear combination of a finite number of Laguerre–Gaussian (LG) modes with the numbers $(n, 0)$:

$$E_{N,-M}\left(r,\varphi,z=0\right)=\exp\left(-\frac{r^2}{w^2}\right)\sum_{n=-M}^{N}C_n\left(\frac{r}{w}\right)^{|n|}e^{in\varphi}. \tag{1.38}$$

Substituting Equation (1.38) into Equation (1.20) yields a relation for TC:

$$TC=\frac{1}{2\pi}\lim_{r\to\infty}\operatorname{Im}\left\{\int_0^{2\pi}i\frac{\sum\limits_{n=-M}^{N}nC_n\left(\frac{r}{w}\right)^{|n|}e^{in\varphi}}{\sum\limits_{n=-M}^{N}C_n\left(\frac{r}{w}\right)^{|n|}e^{in\varphi}}d\varphi\right\}. \tag{1.39}$$

Following a limiting passage $r\to\infty$ under the integral sign in Equation (1.39), the numerator and denominator each retain just one highest-power term under the sum sign. If $M>N$, then TC of the beam in Equation (1.38) is $TC=-M$, if $M<N$, then TC in Equation (1.38) equals $TC=N$. Finally, if $M=N$, instead of Equation (1.39), we obtain:

$$TC=\frac{1}{2\pi}\operatorname{Im}\left\{\int_0^{2\pi}iN\frac{\left(C_Ne^{iN\varphi}-C_{-N}e^{-iN\varphi}\right)}{\left(C_Ne^{iN\varphi}+C_{-N}e^{-iN\varphi}\right)}d\varphi\right\}. \tag{1.40}$$

Thus, we can infer that if in a linear combination of a finite number of LG modes with different TC, the absolute value of the maximum positive TC is larger than the maximum negative TC, the TC of the entire beam equals the positive $TC=N$. If the opposite is the case, the resulting TC equals the negative $TC=-M$. Finally, in the next section we show that for $M=N$, the integral in Equation (1.40) can be taken analytically and, based on Equation (1.42), $TC=N$ if $|C_N|>|C_{-N}|$ or $TC=-N$ if $|C_N|<|C_{-N}|$. When $|C_N|=|C_{-N}|$, TC of the entire beam equals zero.

1.2.5 TC of the Sum of Two Optical Vortices

Now, let us analyze a simple but rather interesting case that produces an unexpected result. Assume a light field with a complex amplitude in the initial plane that describes an axial superposition of two Gaussian OVs with different TC and different amplitudes:

$$E\left(r,\varphi\right)=\left(ae^{in\varphi}+be^{im\varphi}\right)e^{-r^2/w^2}, \tag{1.41}$$

where w is the Gaussian beam waist radius, n and m are integer topological charges of the OVs, a and b are weight coefficients in the OV superposition, which are generally complex. Substituting Equation (1.41) into Equation (1.20) yields a relation for TC:

$$TC = \frac{1}{2\pi}\lim_{r\to\infty}\mathrm{Im}\left\{\int_0^{2\pi}\frac{\partial E(r,\varphi)/\partial\varphi}{E(r,\varphi)}d\varphi\right\} = \frac{1}{2\pi}\mathrm{Re}\left\{\int_0^{2\pi}\frac{nae^{in\varphi}+mbe^{im\varphi}}{ae^{in\varphi}+be^{im\varphi}}d\varphi\right\}. \quad (1.42)$$

The integral in the right-hand side of Equation (1.42) can be reduced to a sum of two integrals:

$$TC = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{n+m}{2}+\frac{n-m}{2}\frac{|a|^2-|b|^2}{|a|^2+|b|^2+2|a||b|\cos t}\right)dt. \quad (1.43)$$

With the first integral in Equation (1.43) being trivial, the second integral can be rearranged as:

$$TC = \frac{n+m}{2}+\frac{1}{2\pi}\frac{n-m}{2}\frac{|a|^2-|b|^2}{|a|^2+|b|^2}\int_0^{2\pi}\frac{dt}{1+\dfrac{2|a||b|}{|a|^2+|b|^2}\cos t}. \quad (1.44)$$

With the coefficient before the cosine function being not larger than unity, this is a reference integral (expression 3.613.1 in [51]):

$$\int_0^{\pi}\frac{\cos(nx)\,dx}{1+a\cos x} = \frac{\pi}{\sqrt{1-a^2}}\left(\frac{\sqrt{1-a^2}-1}{a}\right)^n \quad \left[a^2<1, n\geq 0\right]. \quad (1.45)$$

In the case the integration interval is from zero to 2π , rather than being to π , the expression needs to be multiplied by two. Then, Equation (1.42) takes the form:

$$TC = \frac{n+m}{2}+\frac{n-m}{2}\frac{|a|^2-|b|^2}{\left||a|^2-|b|^2\right|}. \quad (1.46)$$

For completeness sake, the normalized OAM of the beam in Equation (1.41) can be given in the form:

$$OAM = \frac{na^2+mb^2}{a^2+b^2}. \quad (1.47)$$

From Equation (1.46) it follows that if $|a| > |b|$, then $TC=n$ and if $|a|<|b|$, then $TC=m$. If $m=n$, as can be expected, we obtain that $TC=n$. Thus, TC of the resulting beam in Equation (1.41) equals that of the constituent OV with the larger amplitude. At $|a| = |b|$, there occurs degeneracy (photon entanglement), with Equation (1.46) becoming invalid due to uncertainty 0/0. Because of this, at $|a|=|b|$, the field in Equation (1.41) needs to be rearranged to the form:

$$\begin{aligned} E(r,\varphi) &= |a|\left(e^{in\varphi+i\arg a}+e^{im\varphi+i\arg b}\right)e^{-r^2/w^2} \\ &= 2|a|\cos\left(\frac{n\varphi-m\varphi+\arg a-\arg b}{2}\right) \\ &\times\exp\left(-\frac{r^2}{w^2}+i\frac{n\varphi+m\varphi+\arg a+\arg b}{2}\right). \end{aligned} \tag{1.48}$$

Substituting Equation (1.48) into Equation (1.20) yields

$$TC=\lim_{r\to\infty}\frac{1}{2\pi}\int_0^{2\pi}\frac{\partial}{\partial\varphi}\left(\frac{n\varphi+m\varphi+\arg a+\arg b}{2}\right)d\varphi=\frac{n+m}{2}. \tag{1.49}$$

As we can infer from Equation (1.49), the superposition of two same-amplitude OVs, with one *TC* being even and the other odd, produces an OV with a fractional (semi-integer) *TC*. It should be noted that it is only in the initial plane that *TC* of a beam can be fractional, whereas during propagation *TC* needs to be an integer for the amplitude to be continuous. It is worth noting that OAM in Equation (1.47) equals *TC* in Equations (1.46) and (1.49) only when either $a=0$, or $b=0$, or $a=b$. At the same time, if the beam is degenerate ($a=b$), the content of the constituent angular harmonics of the beam cannot be derived from the known *TC*. For instance, all the below-listed beams have the same *TC* and OAM, which is equal to 4:

$$\begin{aligned} E_1(r,\varphi) &= \left(e^{i\varphi}+e^{i7\varphi}\right)e^{-r^2/w^2}, \\ E_2(r,\varphi) &= \left(e^{i2\varphi}+e^{i6\varphi}\right)e^{-r^2/w^2}, \\ E_3(r,\varphi) &= \left(e^{i3\varphi}+e^{i5\varphi}\right)e^{-r^2/w^2}, \\ E_4(r,\varphi) &= e^{i4\varphi}e^{-r^2/w^2}. \end{aligned} \tag{1.50}$$

Figure 1.6(a, b) shows the intensity and phase of the superposition of two Gaussian vortices in the initial plane for the following calculation parameters: waist radius $w=$ 1 mm, topological charges $n=12$ and $m=7$, weight coefficients are unit modulo, but with a random phase: $a=e^{2.9616\,i}$, $b=e^{0.2247\,i}$, computational domain $-R\le x, y\le R$ ($R=$ 1 mm). The *TC* calculated numerically by formula (1.20) is 9.4708, i.e. approximately (12 + 7)/2. Shown in Figure 1.6 (c, d) are the intensity and phase of the same superposition, but at the Fresnel distance (for the wavelength $\lambda=532$ nm) and in a wider calculated region ($R=10$ mm). The *TC* calculated numerically by Equation (1.20) is 11.8167, that is, about 12. In both cases, the *TC* was calculated by integration over a

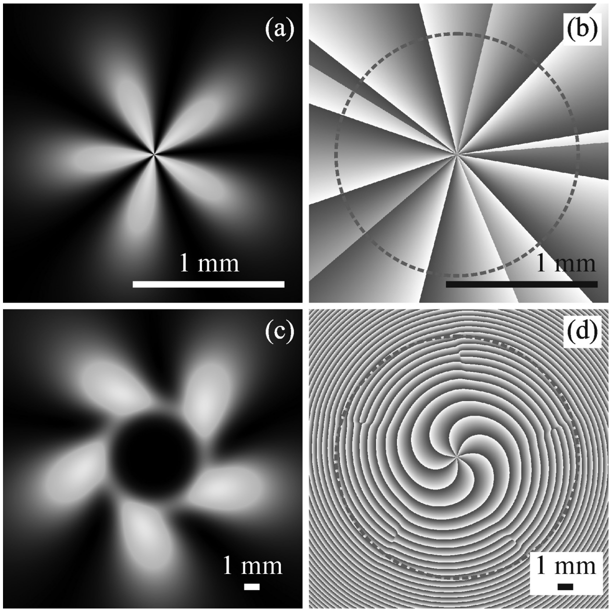


FIGURE 1.6 The intensity (a, c) and phase (b, d) of the axial superposition of two Gaussian OVs with *TC* 12 and 7, but with the same weight amplitudes (in Equation (1.41)) in the initial plane (a, c) and at the Rayleigh distance (c, d). Dashed rings on the phase distributions denote the radius of the ring by which the topological charge was calculated by the Equation (1.20).

ring of radius 0.8*R*. This example corresponds to the situation described by amplitude in Equation (1.46), and when the amplitude moduli are equal $|a|=|b|$ for two vortices, the *TC* of the entire beam will be equal to the larger of the two *TC*, i.e. 12.

1.2.6 Topological Charge in an Arbitrary Plane

In this subsection, we shall demonstrate that a combination of two same-amplitude ($a=b$) Gaussian OVs of Equation (1.41) with different *TC* produces an OV with half-integer *TC* of Equation (1.49) in the initial plane, generating an OV with integer *TC* as it propagates. Actually, if in the initial plane there is a Gaussian OV:

$$E(r,\varphi)=e^{-r^2/w^2+in\varphi}, \tag{1.51}$$

following the propagation through an ABCD-system, its complex amplitude is given by:

$$\begin{aligned}E_z(\rho,\theta)&=(-i)^{n+1}\sqrt{\frac{\pi}{2}}\frac{z_0}{Bq_1}\exp\left(\frac{ikD\rho^2}{2B}+in\theta\right)\\&\times\sqrt{\xi}\exp(-\xi)\left[I_{\frac{n-1}{2}}(\xi)-I_{\frac{n+1}{2}}(\xi)\right],\end{aligned} \tag{1.52}$$

where:

$$\xi = \left(\frac{z_0}{B}\right)^2 \left(\frac{\rho}{w}\right)^2 \left(\frac{1}{2q_1}\right), q_1 = 1 - i\frac{A}{B} z_0. \tag{1.53}$$

Since in Equation (1.52), $I_\nu\,(x)$ is a modified Bessel function, then for the superposition in Equation (1.41), the complex amplitude is:

$$\begin{aligned} E_z(\rho,\theta) = -i\sqrt{\frac{\pi}{2}}\frac{z_0}{Bq_1}\exp\left(\frac{ikD\rho^2}{2B}\right)\sqrt{\xi}\exp(-\xi)\times \\ \times\left\{a(-i)^n \exp(in\theta)\left[I_{\frac{n-1}{2}}(\xi) - I_{\frac{n+1}{2}}(\xi)\right]\right. \\ \left. + b(-i)^m \exp(im\theta)\left[I_{\frac{m-1}{2}}(\xi) - I_{\frac{m+1}{2}}(\xi)\right]\right\}. \end{aligned} \tag{1.54}$$

Retaining in the asymptotic expansion of the modified Bessel function just first two terms yields a relationship to describe the difference of two modified Bessel functions of adjacent orders at large values of the argument:

$$I_{\frac{n-1}{2}}(\xi) - I_{\frac{n+1}{2}}(\xi) \sim \frac{e^\xi}{\sqrt{2\pi\xi}}\left\{\left[1 - \frac{4\left(\frac{n-1}{2}\right)^2 - 1}{8\xi}\right] - \left[1 - \frac{4\left(\frac{n+1}{2}\right)^2 - 1}{8\xi}\right]\right\} = \frac{ne^\xi}{2\xi\sqrt{2\pi\xi}}. \tag{1.55}$$

Then, at large values of ρ , Equation (1.54) takes the form:

$$\begin{aligned} E_z(\rho,\theta) = -i\sqrt{\frac{\pi}{2}}\frac{z_0}{Bq_1}\exp\left(\frac{ikD\rho^2}{2B}\right)\sqrt{\xi}\exp(-\xi) \\ \times\left[a(-i)^n \exp(in\theta)\frac{ne^\xi}{2\xi\sqrt{2\pi\xi}} + b(-i)^m \exp(im\theta)\frac{me^\xi}{2\xi\sqrt{2\pi\xi}}\right] \\ = \frac{-iz_0}{4Bq_1\xi}\exp\left(\frac{ikD\rho^2}{2B}\right)\left[an(-i)^n \exp(in\theta) + bm(-i)^m \exp(im\theta)\right]. \end{aligned} \tag{1.56}$$

In view of Equations (1.42) and (1.46), Equation (1.56) suggests that at $|a| = |b|$, in the initial plane $TC = (m + n)/2$. At the same time, in any other plane, modules of the coefficients in front of $e^{in\theta}$ and $e^{im\theta}$ are proportional to $|n|$ and $|m|$, being no more equal to each other (at $n \neq m$), hence, according to Equation (1.46), $TC = \max(n, m)$.

Note, however, that if in Equation (1.56) $|an| = |bm|$, we again find ourselves in the situation of degeneracy, because in view of Equation (1.46) and at $z > 0$, TC of two OVs of Equation (1.41) equals the arithmetic mean of Equation (1.49): $TC = (n + m)/2$. This situation can be addressed as follows: with the equality $|an| = |bm|$ meaning that $|a| \neq |b|$, Equation (1.46) suggests that the total TC of the field in the initial plane equals that of the OV with the larger amplitude (respectively, $|a|$ or $|b|$). In the meantime, the integer TC in the initial plane conserves upon propagation.

1.2.7 Topological Charge for an Optical Vortex with an Initial Fractional Charge

For an OV with fractional $TC = \mu$ (μ is an arbitrary real number), a relation to describe the corresponding fractional TC has been derived [52]. The mutual transformations between beams with fractional-order and integer-order vortices were considered in detail in [53]. An OV with fractional TC, which is possible only in the initial plane, can be decomposed in terms of OVs with integer TC n (μ is an arbitrary real number) as follows:

$$E_\mu(r,\varphi,z) = \exp(-i\mu\varphi)\Psi(r,z) = \frac{e^{i\pi\mu}\sin\pi\mu}{\pi}\Psi(r,z)\sum_{n=-\infty}^{\infty}\frac{e^{in\varphi}}{\mu-n}. \tag{1.57}$$

In Equation (1.57), the function $\Psi(r, z)$ is real. Substituting the right-hand side of Equation (1.57) into a general relation for OAM:

$$J_z = \mathrm{Im}\int_0^\infty\int_0^{2\pi}\bar{E}(r,\varphi,z)\left(\frac{\partial E(r,\varphi,z)}{\partial\varphi}\right)rdrd\varphi \tag{1.58}$$

yields:

$$J_z = W\frac{\sin^2(\pi\mu)}{\pi^2}\sum_{n=-\infty}^{\infty}\frac{n}{(\mu-n)^2}, \tag{1.59}$$

where W is the energy (power) of the beam:

$$W = \int_0^\infty\int_0^{2\pi}E(r,\varphi,z)\bar{E}(r,\varphi,z)rdrd\varphi. \tag{1.60}$$

The series in the right-hand side of Equation (1.59) can be reduced to a reference series [51]:

$$\sum_{n=1}^{\infty}\frac{n^2}{\left(n^2\pm a^2\right)^2} = \frac{\pi}{4a}\left[\pm\begin{Bmatrix}\coth\pi a\\ \cot\pi a\end{Bmatrix}\mp a\begin{Bmatrix}\mathrm{cosech}^2\,\pi a\\ \mathrm{cosec}^2\,\pi a\end{Bmatrix}\right], \tag{1.61}$$

using which, the final relation for the normalized OAM of the field in Equation (1.59) is rearranged to:

$$\frac{J_z}{W} = \mu - \frac{\sin 2\pi\mu}{2\pi}. \tag{1.62}$$

From Equation (1.62) it follows that OAM equals $TC = \mu$ only if μ is integer and half-integer. This conclusion is in agreement with Equations (1.53) and (1.55) for the linear combination composed of two angular harmonics.

We obtain the expression for the TC of the optical vortex in the Fresnel diffraction zone for the initial field with a fractional topological charge from Equation (1.57), but for definiteness we choose the amplitude function in the form of a Gaussian one. Then instead of Equation (1.57) we get:

$$E_\mu(r,\varphi,z=0) = \exp(-i\mu\varphi - \left(\frac{r}{w}\right)^2) = \frac{e^{i\pi\mu}\sin\pi\mu}{\pi}\sum_{n=-\infty}^{\infty}\frac{e^{in\varphi - r^2/w^2}}{\mu - n}. \tag{1.63}$$

In view of Equation (1.52), the amplitude of the optical vortex in Equation (1.63) for any z will be equal to ($B = z$, $A = D = 1$):

$$\begin{aligned} E_2(\rho,\theta) &= \frac{1}{\sqrt{2\pi}}\left(\frac{-iz_0}{q_1 z}\right)\exp\left(\frac{ik\rho^2}{2z} + i\pi\mu\right)\sin(\pi\mu)\sqrt{x}\exp(-x) \\ &\times \sum_{m=-\infty}^{\infty}(-i)^m(\operatorname{sgn} m)^{|m|}\frac{\exp(im\theta)}{\mu - m}\left[I_{\frac{|m|-1}{2}}(x) - I_{\frac{|m|+1}{2}}(x)\right] \end{aligned} \tag{1.64}$$

We substitute Equation (1.64) in Equation (1.20) and, when passing to the limit in Equation (1.20), we take into account the asymptotic behavior in Equation (1.36), then we obtain the expression for calculating the TC of the optical vortex (1.63):

$$TC = \frac{\operatorname{Re}}{2\pi}\left\{\int_0^{2\pi}\frac{\sum_{n=-\infty}^{\infty}\frac{(-i)^{|n|}n|n|e^{in\varphi}}{\mu - n}}{\sum_{n=-\infty}^{\infty}\frac{(-i)^{|n|}|n|e^{in\varphi}}{\mu - n}}d\varphi\right\}. \tag{1.65}$$

Equation (1.65) is remarkable in that the answer is known, which was numerically obtained in [19], but has not yet been obtained analytically. Calculation (1.65) can be called the Berry problem [19]. The right-hand side of Equation (1.65) should give only whole TC, closest to μ :

$$TC = \sum_{n=-\infty}^{\infty} n\,\operatorname{rect}(\mu - n), \quad \operatorname{rect}(x) = \begin{cases} 1, |x| \le 1/2, \\ 0, |x| > 1/2 \end{cases} \tag{1.66}$$

From a comparison of Equations (1.65) and (1.66), it can be said that the *TC* in Equation (1.65) is equal to the *TC* of the angular harmonic in the series in the numerator and the denominators for which the weight coefficient is greater in absolute value. This is also consistent with the results for a complete linear combination of LG modes in Equation (1.38) and for the sum of two angular harmonics in Equation (1.41).

The *TC* of an optical vortex can be measured using a cylindrical lens by the method described in [15]. Figure 1.7 shows the intensity distributions at a double focal length from a cylindrical lens for optical vortices with an initial fractional *TC* in Equation (1.57). It can be seen that on the line at an angle of –45 degrees in the center of the picture are two zeros (two dark lines) (Figure 1.7a) for $\mu < 2.5$ and three zeros (three dark lines) for $\mu > 2.5$ (Figure 1.7b, c, d). As shown in Figure 1.7, for arbitrary initial fractional *TC* between 2 and 2.5, *TC* of the optical vortex equals 2, and for arbitrary initial fractional *TC* (Equation (1.57)) higher than 2.5 and lower than 3, *TC* of the beam equals 3. The experiment in Figure 1.7 confirms the numerical result in Equation (1.66).

1.2.8 Topological Charge of an Elliptic Optical Vortex Embedded in a Gaussian Beam

Let us analyze a simple example of an OV with introduced phase distortion by making it ellipse-shaped. While for a conventional OV the complex amplitude in the initial plane is given by:

$$E(r,\varphi) = A(r)\exp(in\varphi), \tag{1.67}$$

for an elliptic vortex imbedded, say, into a Gaussian beam (or any other radially symmetric beam) it takes the form:

$$\begin{aligned} E_e(x,y) &= A(\sqrt{x^2+y^2})(x+i\alpha y)^n \\ &= A(\sqrt{x^2+y^2})(x^2+\alpha^2 y^2)^{n/2}\exp\left(in\arctan\left(\frac{\alpha y}{x}\right)\right). \end{aligned} \tag{1.68}$$

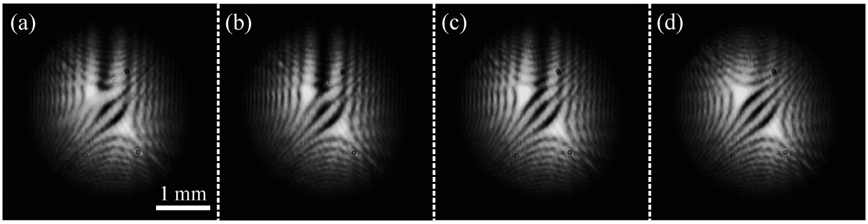


FIGURE 1.7 Intensity distributions measured at a distance $z = 200$ mm (at a double focal length from a cylindrical lens) from a spiral phase plate with fractional order μ : (a) 2.3, (b) 2.5, (c) 2.7, (g) 2.9. The sizes of the images are 4,000 by 4,000 microns.

Substituting Equation (1.68) into Equation (1.20) yields:

$$TC = \frac{1}{2\pi}\int_0^{2\pi} d\varphi \frac{\partial}{\partial\varphi} \arg E_e(r,\varphi) = \frac{1}{2\pi}\int_0^{2\pi} d\varphi \frac{\partial}{\partial\varphi}\left(n \arctan\left(\alpha \tan\varphi\right)\right)$$
$$= \left(\frac{n\alpha}{2\pi}\right)\int_0^{2\pi} \frac{d\varphi}{\cos^2\varphi + \alpha^2 \sin^2\varphi} = n. \tag{1.69}$$

Note that a result similar to Equation (1.69), but only for $n = 1$, was previously obtained in [54]. From Equation (1.69) it follows that the fact that an optical vortex or SPP is ellipse-shaped does not change the TC of the original simple OV in Equation (1.67). At any degree of ellipticity (any α), an elliptic OV has $TC = n$. In the meantime, OAM of an elliptic OV is always lower than n, being equal to:

$$\frac{J_z}{W} = \frac{nP_{n-1}\left(y\right)}{P_n\left(y\right)} < n, \tag{1.70}$$

where $y = (1 + \alpha^2)/(2\alpha) > 1$ and $P_n(y)$ is Legendre polynomial. Figure 1.8 shows the intensity and phase distributions of a Gaussian beam with an elliptical vortex in the initial plane and after propagation in space for different ellipticities. The complex amplitude in the initial plane is $E_e\left(x,y\right) = \exp\left[-\left(x^2+y^2\right)/w^2\right]\left(x+i\alpha y\right)^n$, where w is the waist radius of the Gaussian beam, n and α are the topological charge and ellipticity of the optical vortex, respectively. The following calculation parameters were used: $w = 1$ mm, $n = 7$, $\alpha = 1.1$ (Figure 1.8(a,b,c,d)), $\alpha = 1.5$ (Figure 1.8(e,f,g,h)), $\alpha = 3$ (Figure 1.8(i,j,r,l)), the distance of propagation in space $z = z_0/2$, the computational domain is $-R \le x,\ y \le R$ ($R = 5$ mm). The TC in the initial plane, calculated numerically by Equation (1.20) (along a ring of radius $0.8R$), is 6.9997 at $\alpha = 1.1$, 6.9996 at $\alpha = 1.5$, 6.9987 at $\alpha = 3$, that is, in all cases about 7. TC turned out to be 6.9989, 6.9988, and 6.9979, respectively. That is, it is also approximately equal to 7.

Summing up, it has been theoretically shown [55] that OVs conserve the integer TC when passing through an arbitrary aperture or shifted from the optical axis of an arbitrary axisymmetric carrier beam. If the beam contains a finite number of off-axis optical vortices with different-value same-sign TC, the total TC of the resulting beam has been shown to be equal to the sum of all constituent TCs. In this case, there is no topological competition, because it takes place as a result of on-axis superposition of OVs. By way of illustration, if an on-axis superposition is composed of a finite number of Laguerre–Gaussian modes (n, 0), the resulting TC equals that of the constituent mode with the highest TC (including sign). If the highest positive and negative TCs of the constituent modes are equal in magnitude, the "winning" TC is the one with the larger absolute value of the weight coefficient. If the constituent modes have the same weight coefficients, the resulting TC equals zero. If the beam is composed of two on-axis different-amplitude Gaussian vortices with different TC, the resulting TC equals that of the constituent vortex with the larger absolute

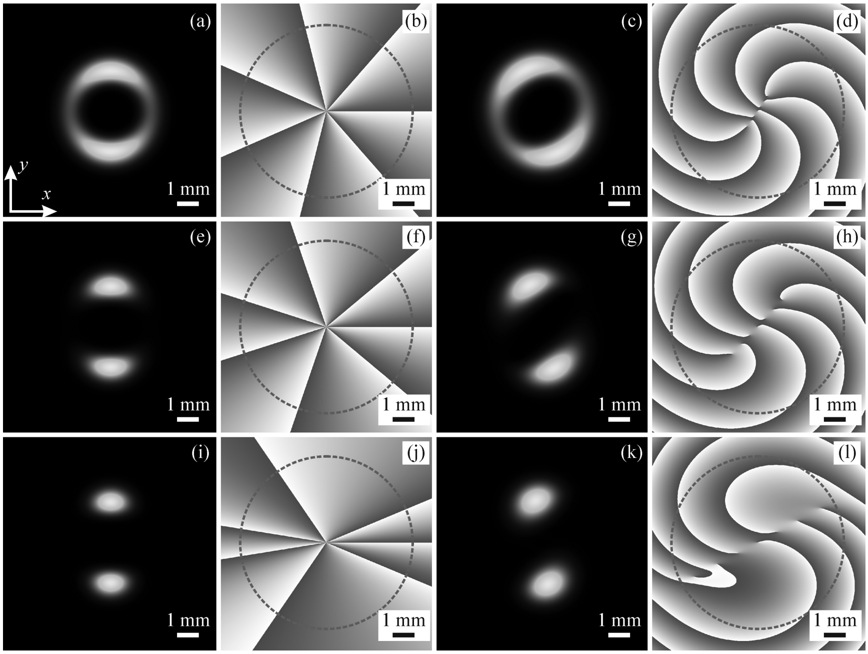


FIGURE 1.8 Distributions of intensity (a, c, e, g, i, k) and phase (b, d, f, h, j, l) of a Gaussian beam with an elliptical vortex in the initial plane (a, b, e, f, i, j) and after propagation in space (c, d, g, h, k, l) for different ellipticities. Dashed rings on the phase distributions denote the radius of the ring by which the *TC* was calculated by the Equation (1.20).

value of the weight coefficient amplitude, irrespective of the correlation between the individual TCs. If the constituent beams have equal weight coefficients, there occurs degeneracy, with the resulting *TC* being equal to the mean arithmetic of the constituent Gaussian OVs. If in the superposition of two Gaussian OVs, one *TC* is odd and the other is even, the resulting *TC* in the initial plane is half-integer. As the beam propagates, degeneracy is eliminated, with the resulting *TC* becoming equal to the larger (positive) integer constituent *TC*. This effect has been given the name "topological competition of optical vortices". Theoretical predictions have been corroborated by numerical simulation and experiments.

1.3 TOPOLOGICAL CHARGE OF ASYMMETRIC OPTICAL VORTICES

Presently, laser vortex beams [56], or optical vortices (OV), have been actively studied because they have found uses in many optical applications. For instance, OVs are utilized in quantum information science [57], cryptography [58], wireless communication systems [59], data transmission in optical fibers [60], second-harmonic generation [61], short-pulse interferometry [62], and probing of turbulent media

[63]. Vortex beams are characterized by two major parameters, namely, topological charge (*TC*) [19] and orbital angular momentum (OAM) [33], which describe different aspects of an OV. With *TC* depending only on the phase of a light field, OAM is both phase- and amplitude- (intensity) dependent. *TC* can be measured using a cylindrical lens [15] or a triangular aperture [64]. For measuring OAM, a cylindrical lens can also be utilized [12,13]. The OAM spectrum of OVs, which defines the energy contribution in each constituent angular harmonic of the laser beam, can be measured with a multi-order diffractive optical element [65] or based on intensity moments [66,11]. For radially symmetric OVs (e.g. LG and BG beams [4,40]), whose complex amplitude can be given by $E(r,\varphi,z) = A(r,z)\exp(in\varphi)$, where $A(r, z)$ is the radial component of the beam's complex amplitude, n is *TC* of the beam, and (r, φ, z) are the cylindrical coordinates, *TC* is defined by OAM normalized to the beam's power and equals n. It is worth noting that an integer *TC* of a radially symmetric OV remains unchanged upon propagation. For other types of vortex beams, *TC* needs to be calculated individually. Meanwhile OAM of the beam remains unchanged upon propagation and can be calculated in the source plane, for *TC* this is not always the case. For instance, *TC* of a combined beam composed of two LG modes with different-waist radii is not conserved [22].

In this section, we derive relationships to define *TC* of certain radially asymmetric vortex laser beams. In previous publications of the present authors, normalized OAMs of such beams were derived, but patterns of *TC* behavior were not analyzed. Below, we derive relationships to describe *TC* of asymmetric LG, BG, and Kummer beams [44,67,68], superposition of two HG modes [69], and a vortex HG beam [70]. We note that the considered asymmetrical beams are obtained in different ways. Asymmetric LG and Kummer beams are obtained from the conventional symmetric LG and Kummer beams by a transverse complex shift in the Cartesian coordinates. Asymmetric BG beams are obtained by a hybrid technique: complex shift is applied only to the Bessel function, whereas the Gaussian function remains unshifted. Vortex HG beams are derived from the conventional HG beam by an astigmatic transform using a cylindrical lens. And another vortex beam is a superposition of two HG modes with complex weight coefficients. Therefore, we calculated *TC* for each type of beam separately.

1.3.1 TC of an Asymmetric LG Beam

Upon free-space propagation of an asymmetric LG (aLG) beam, at a distance z its complex amplitude is given by [44]:

$$\begin{aligned} E(x,y,z) &= \frac{w(0)}{w(z)}\left[\frac{\sqrt{2}}{w(z)}\right]^{|l|}\left[(x-x_0)+i\theta(l)(y-y_0)\right]^{|l|} \\ &\times L_p^{|l|}\left[\frac{2\rho^2}{w^2(z)}\right]\exp\left[-\frac{\rho^2}{w^2(z)}+\frac{ik\rho^2}{2R(z)}-i\left(|l|+2p+1\right)\zeta(z)\right], \end{aligned} \tag{1.71}$$

where:

$$\begin{aligned}
&\rho^2 = \left(x - x_0\right)^2 + \left(y - y_0\right)^2, \\
&w\left(z\right) = w\sqrt{1 + \left(\frac{z}{z_R}\right)^2}, \\
&R\left(z\right) = z\left[1 + \left(\frac{z_R}{z}\right)^2\right], \\
&\zeta\left(z\right) = \arctan\left(\frac{z}{z_R}\right),
\end{aligned} \tag{1.72}$$

where $\theta(l) = \left\{1, l \geq 0; -1, l < 0\right\}$, (x, y, z) and (r, φ, z) are the Cartesian and cylindrical coordinates, (x_0, y_0) are the complex coordinates of the off-axis shift of the LG beam, w is the Gaussian beam waist radius, l is *TC* of the optical vortex, $L_p^l\left(x\right)$ is the associated Laguerre polynomial, $z_R = kw^2/2$ is the Rayleigh range, and $k = 2\pi/\lambda$ is the wavenumber of light of wavelength λ. The transverse intensity of the beam is not radially symmetric, unlike conventional LG beams [4]. If (x_0, y_0) are real, beam in Equation (1.71) becomes a conventional off-axis LG mode.

Below, we discuss the TCs of various OVs derived in this work in relation to their OAMs derived in the previous studies. In doing so, we shall make use of formulae for calculating the OAM of paraxial laser beams and beam power [44]:

$$J_z = \operatorname{Im}\iint_{\mathbb{R}^2} E^*\left(x\frac{\partial E}{\partial y} - y\frac{\partial E}{\partial x}\right)dxdy, \tag{1.73}$$

$$W = \iint_{\mathbb{R}^2} E^* E dxdy. \tag{1.74}$$

For an an LG beam, OAM normalized to power is given by [44]:

$$\frac{J_z}{W} = l + \frac{2\operatorname{Im}\left(x_0^* y_0\right)}{w^2}\left[\frac{L_p^1\left(\dfrac{Q^2}{2w^2}\right)}{L_p\left(\dfrac{Q^2}{2w^2}\right)} + \frac{L_{p+l}^1\left(\dfrac{Q^2}{2w^2}\right)}{L_{p+l}\left(\dfrac{Q^2}{2w^2}\right)} - 1\right]. \tag{1.75}$$

where:

$$Q = 2i\sqrt{\left(\operatorname{Im} x_0\right)^2 + \left(\operatorname{Im} y_0\right)^2}. \tag{1.76}$$

Unlike *TC*, an increase or decrease in OAM is fully determined by the sign of the quantity $\mathrm{Im}\left(x_0^* y_0\right)$, because the relation in square brackets in Equation (1.75) is always larger than or equal to the unit.

Below, we calculate the *TC* of an aLG beam of Equation (1.71), using Berry's formula [19]:

$$TC = \lim_{r\to\infty} \frac{1}{2\pi} \int_0^{2\pi} d\varphi \frac{\partial}{\partial\varphi} \arg E\left(r,\varphi\right) = \frac{1}{2\pi} \lim_{r\to\infty} \mathrm{Im} \int_0^{2\pi} d\varphi \frac{\partial E\left(r,\varphi\right)/\partial\varphi}{E\left(r,\varphi\right)}. \tag{1.77}$$

We note that the standard definition of the *TC* is the number of 2π phase changes on a closed loop. Unfortunately, this definition is not constructive, and does not allow analytical calculation of the *TC* of optical vortices. It is a great merit of Berry that he proposed a more constructive *TC* in Equation (1.77), which we use in this paper. Both of these definitions lead to the same result.

Assuming ($l > 0$) that the complex shift in Equation (1.72) is given by $x_0 = aw$, $y_0 = iaw$, the term $[(x-x_0)+i(y-y_0)]^l$ in Equation (1.71) takes a simple form $r^l e^{il\varphi}$, with the variable ρ^2 in Equation (1.72) taking the following form: $\rho^2 = (x-x_0)^2+(y-y_0)^2 = r^2-2awre^{i\varphi}$, where a is a dimensionless constant whose magnitude defines the asymmetry of the beam. With due regard to the above considerations, the derivative with respect to the angle φ of the function in Equation (1.71) takes the form:

$$\begin{aligned}\frac{\partial E\left(r,\varphi,z\right)}{\partial\varphi} &= ilE\left(r,\varphi,z\right)\\ &-\left[-\frac{1}{w^2\left(z\right)}+\frac{ik}{2R\left(z\right)}\right]\left(2iawre^{i\varphi}\right)E\left(r,\varphi,z\right)\\ &-\frac{4iawre^{i\varphi}}{w^2\left(z\right)}\frac{1}{L_p^{|l|}\left(\xi\right)}\frac{d}{d\xi}L_p^{|l|}\left(\xi\right)E\left(r,\varphi,z\right),\end{aligned} \tag{1.78}$$

where $\xi = \dfrac{2\rho^2}{w^2\left(z\right)}$

Substituting Equation (1.78) into Equation (1.77) yields ($w = w(0)$):

$$\begin{aligned}TC &= \frac{1}{2\pi}\lim_{r\to\infty}\mathrm{Im}\int_0^{2\pi} d\varphi\left\{il-\left[-\frac{1}{w^2(z)}+\frac{ik}{2R(z)}\right]\left(2iarwe^{i\varphi}\right)-\frac{i4awre^{i\varphi}}{w^2(z)}\frac{1}{L_m^{|l|}(\xi)}\frac{\partial L_m^{|l|}\left(\xi\right)}{\partial\xi}\right\}\\ &= l+\lim_{r\to\infty}\mathrm{Re}\left[\left(-\frac{2}{w^2(z)}+\frac{ik}{R(z)}\right)raw\int_0^{2\pi} d\varphi e^{i\varphi}\right]-\lim_{r\to\infty}\mathrm{Re}\int_0^{2\pi} d\varphi\frac{2r\left|l\right|awe^{i\varphi}}{r\left(r-2awe^{i\varphi}\right)} = l.\end{aligned} \tag{1.79}$$

References

J. Durnin , J.J. Micely , and J.H. Eberly , “Diffraction-free beams,” Phys Rev Lett. 58(15), 1499–1501 (1987).

M.A. Bandres , J.C. Gutiérrez-Vega , and S. Chávez-Cerda , “Parabolic nondiffracting optical wave fields,” Opt Lett. 29(1), 44–46 (2004).

J.C. Gutiérrez-Vega , M.D. Iturbe-Castillo , and S. Chávez-Cerda , “Alternative formulation for invariant optical fields: Mathieu beams,” Opt Lett. 25(20), 1493–1495 (2000).

A.E. Siegman , Lasers (University Science Book, Mill Valley, 1986).

N. Efremidis , Z. Chen , M. Segev , and D. Christodoulides , “Airy beams and accelerating waves: an overview of recent advances,” Optica 6(5), 686 (2019).

P. Zhang , Y. Hu , T. Li , D. Cannan , X. Yin , R. Morandotti , Z. Chen , and X. Zhang , “Nonparaxial Mathieu and Weber Accelerating Beams,” Phys Rev Lett. 109(19), 193901 (2012).

J. Turunen , and A.T. Friberg , “Propagation-invariant optical fields,” Prog Opt. 54, 1–88 (2010).

U. Levy , S. Derevyanko , and Y. Silberberg , “Light modes of free space,” Prog Opt. 61, 237–281 (2016).

H. Xiong , Y. Huang , and Y. Wu , “Laguerre-Gaussian optical sum-sideband generation via orbital angular momentum exchange,” Phys. Rev. A 103(4), 043506 (2021).

G. Gbur , and R.K. Tuson , “Vortex beam propagation through atmospheric turbulence and topological charge conservation,” J. Opt. Soc. Am. A 25(1), 225–230 (2008).

A.V. Volyar , M.V. Brezko , Ya.E. Akimova , Yu.A. Egorov , and V.V. Milyukov , “Sector perturbation of a vortex beam: Shannon entropy, orbital angular momentum and topological charge,” Comput Opt. 43(5), 723–734 (2019).

S.N. Alperin , R.D. Niederriter , J.T. Gopinath , and M.E. Siemens , “Quantitative measurement of the orbital angular momentum of light with a single, stationary lens,” Opt Lett. 41(21), 5019–5022 (2016).

V.V. Kotlyar , A.A. Kovalev , and A.P. Porfirev , “Calculation of fractional orbital angular momentum of superpositions of optical vortices by intensity moments,” Opt. Express 27(8), 11236–11251 (2019).

L.A. Melo , A.J. Jesus-Silva , S. Chávez-Cerda , P.H. Souto Ribeiro , and W.C. Soares , “Direct measurement of the topological charge in elliptical beams using diffraction by a triangular aperture,” Sci Rep. 8(1), 6370 (2018).

V.V. Kotlyar , A.A. Kovalev , and A.P. Pofirev , “Astigmatic transforms of an optical vortex for measurement of its topological charge,” Appl Opt. 56(14), 4095–4104 (2017).

L. Allen , S.M. Barnett , and M.J. Padgett , Orbital angular momentum (CRC Press, Boca Raton, 2003).

J. Courtial , K. Dholakia , L. Allen , and M.J. Padgett , “Gaussian beams with very high orbital angular momentum,” Opt Commun. 144(4–6), 210–213 (1997).

I. Martinez-Castellanos and J. Gutiérrez-Vega , “Shaping optical beams with non-integer orbital-angular momentum: A generalized differential operator approach,” Opt Lett. 40(8), 1764–1767 (2015).

M.V. Berry , “Optical vortices evolving from helicoidal integer and fractional phase steps,” J Opt A: Pure Appl Opt. 6(2), 259–268 (2004).

L. Allen , M.J. Padgett , and M. Babiker , “IV The orbital angular momentum of light,” Prog Opt. 39, 291–372 (1999).

J.F. Nye , and M.V. Berry , “Dislocations in wave trains,” Proc. R. Soc. Lond. A 336, 165–190 (1974).

M.S. Soskin , V.N. Gorshkov , M.V. Vastnetsov , J.T. Malos , and N.R. Heckenberg , “Topological charge and angular momentum of light beams carrying optical vortex,” Phys. Rev. A 56(5), 4064–4075 (1997).

V.V. Kotlyar , A.A. Kovalev , and A.G. Nalimov , “Optical phase singularities ‘going to' infinity with a higher-than-light speed,” J Opt. 23(10), 105702 (2021).

F. Gori , in Current trends in optics, ed. by J. C. Dainty (Academic Press, London, 1994), p. 140.

J. Humblet , “Sur le moment d'impulsion d'une onde electromagnetique,” Physica 10(7), 585 (1943).

Sommerfeld , Lectures on theoretical physics (Academic Press, New York, 1954), pp. 361–373.

Z. Sacks , D. Rozas , and G. Swartzlander , “Holographic formation of optical-vortex filaments,” J. Opt. Soc. Am. B 15(8), 2226 (1998).

V.V. Kotlyar , A.A. Kovalev , R.V. Skidanov , O.Yu. Moiseev , and V.A. Soifer , "Diffraction of a finite-radius plane wave and a Gaussian beam by a helical axicon and a spiral phase plate," J. Opt. Soc. Am. A 24(7), 1955 (2007).

M.R. Dennis , "Rows of optical vortices from elliptically pertubing a high-order beam," Opt Lett. 31(9), 1325–1327 (2006).

A.A. Kovalev , V.V. Kotlyar , and A.G. Nalimov , "Topological charge and asymptotic phase invariants of vortex laser beams," Photonics 8(10), 445 (2021).

B. Paroli , M. Siano , and M. Potenza , "Measuring the topological charge of orbital angular momentum radiation in single-shot by means of the wavefront intrinsic curvature," Appl Opt. 59(17), 5258 (2020).

D. Wang , H. Huang , H. Toyoda , and H. Liu , "Topological Charge Detection Using Generalized Contour-Sum Method from Distorted Donut-Shaped Optical Vortex Beams: Experimental Comparison of Closed Path Determination Methods," Appl Sci. 9(19), 3956 (2019).

L. Allen , M. Beijersbergen , R. Spreeuw , and J. Woerdman , "Orbital angular momentum of light and the transformation of Laguerre-Gaussian laser modes," Phys. Rev. A 45(11), 8185 (1992).

A. Volyar , M. Bretsko , Ya. Akimova , and Yu. Egorov , "Vortex avalanche in the perturbed singular beams," J. Opt. Soc. Am. A 36(6), 1064–1071 (2019).

Y. Zhang , X. Yang , and J. Gao , "Orbital angular momentum transformation of optical vortex with aluminium metasurfaces," Sci Rep. 9(1), 9133 (2019).

H. Zhang , X. Li , H. Ma , M. Tang , H. Li , J. Tang , and Y. Cai , "Grafted optical vortex with controllable orbital angular momentum distribution," Opt. Express 27(16), 22930–22938 (2019).

A. Volyar , M. Bretsko , Ya. Akimova , and Yu. Egorov , "Measurement of the vortex and orbital angular momentum spectra with a single cylindrical lens," Appl Opt. 58(21), 5748–5755 (2019).

V.V. Kotlyar , A.A. Kovalev , A.P. Porfirev , and E.S. Kozlova , "Orbital angular momentum of a laser beam behind an off-axis spiral phase plate," Opt Lett. 44(15), 3673–3676 (2019).

S. Maji , P. Jacob , and M.M. Brundavanam , "Geometric phase and intensity-controlled extrinsic orbital angular momentum of off-axis vortex beams," Phys Rev Appl. 12(5), 054053 (2019).

F. Gori , G. Guattary , and C. Padovani , "Bessel-Gauss beams," Opt Commun. 64(6), 491–495 (1987).

V.V. Kotlyar , R.V. Skidanov , S.N. Khonina , and V.A. Soifer , "Hypergeometric modes," Opt Lett. 32(7), 742–744 (2007).

M.A. Bandres , and J.C. Gutierrez-Vega , "Circular beams," Opt Lett. 33(2), 177–179 (2008).

V.V. Kotlyar , A.A. Kovalev , and V.A. Soifer , "Asymmetric Bessel modes," Opt Lett. 39(8), 2395–2398 (2014).

A.A. Kovalev , V.V. Kotlyar , and A.P. Porfirev , "Asymmetric Laguerre-Gaussian beams," Phys. Rev. A 93(6), 063858 (2016).

M.A. Garcia-March , A. Ferrando , M. Zacares , S. Sahu , and D.E. Ceballos-Herrera , "Symmetry, winding number, and topological charge of vortex solitons in discrete-symmetry media," Phys. Rev. A 79(5), 053820 (2009).

A. Volyar , M. Bretsko , Ya. Akimova , and Yu. Egorov , "Orbital angular momentum and informational entropy in perturbed vortex beams," Opt Lett. 44(22), 2687–2690 (2019).

V.V. Kotlyar , A.A. Kovalev , and A.P. Porfirev , "Asymmetric Gaussian optical vortex," Opt Lett. 42(1), 139–142 (2017).

V.V. Kotlyar , A.A. Kovalev , A.P. Porfirev , and E.G. Abramochkin , "Fractional orbital angular momentum of a Gaussian beam with an embedded off-axis optical vortex," Comput Opt. 41(1), 22–29 (2017).

G. Indebetouw , "Optical vortices and their propagation," J Mod Opt. 40(1), 73–87 (1993).

E. Abramochkin , and V. Volostnikov , "Spiral-type beams," Opt Commun. 102(3–4), 336–350 (1993).

I.S. Gradshteyn , and I.M. Ryzhik , Table of integrals, series, and products (Academic, New York, 1965).

J.B. Gotte , S. Franke-Arnold , R. Zambrini , and S.M. Barnett , "Quantum formulation of fractional orbital angular momentum," J Mod Opt. 54(12), 1723–1738 (2007).

C.N. Alexeyev , Yu.A. Egorov , and A.V. Volyar , "Mutual transformations of fractional-order and integer-order optical vortices," Phys. Rev. A 96(6), 063807 (2017).

G. Liang and W. Cheng , “Splitting and rotating of optical vortices due to non-circular symmetry in amplitude and phase distributions of the host beams,” Phys. Lett. A 384, 126046 (2020).
V.V. Kotlyar , A.A. Kovalev , and A.V. Volyar , “Topological charge of a linear combination of optical vortices: Topological competition,” Opt. Express 28(6), 8266–8281 (2020).
V.V. Kotlyar , A.A. Kovalev , and A.P. Porfirev , Vortex laser beams (CRC Press, Boca Raton, 2018).
S. Li , X. Pan , Y. Ren , H. Liu , S. Yu , and J. Jing , “Deterministic generation of orbital-angular-momentum multiplexed tripartite entanglement,” Phys Rev Lett. 124(8), 083605 (2020).
M. Hiekkamaki , S. Prabhakar , and R. Fickler , “Near-perfect measuring of full-field transverse-spatial modes of light,” Opt. Express 27(22), 31456–31464 (2019).
S. Li , X. Li , L. Zhang , G. Wang , L. Zhang , M. Liu , C. Zeng , L. Wang , Q. Sun , W. Zhao , and W. Zhang , “Efficient optical angular momentum manipulation for compact multiplexing and demultiplexing using a dielectric metasurface,” Adv Opt Mater. 2020, 1901666 (2020).
A. Pryamikov , G. Alagashev , G. Falkovich , and S. Turitsyn , “Light transport and vortex-supported wave-guiding in micro-strustured optical fibers,” Sci Rep. 10(1), 2507 (2020).
K. Dai , W. Li , K.S. Morgan , Y. Li , J.K. Miller , R.J. Watkins , and E.G. Johnson , “Second-harmonic generation of asymmetric Bessel-Gaussian beams carrying orbital angular momentum,” Opt. Express 28(2), 2536–2546 (2020).
N. Dimitrov , M. Zhekova , G.G. Paulus , and A. Dreischuh , “Inverted field interferometer for measuring the topological charges of optical vortices carried by short pulses,” Opt Commun. 456, 124530 (2020).
R.J. Watkins , K. Dai , G. White , W. Li , J.K. Miller , K.S. Morgan , and E.G. Johnson , “Experimental probing of turbulence using a continuous spectrum of asymmetric OAM beams,” Opt. Express 28(2), 924–935 (2020).
J.M. Hickmann , E.J.S. Fonseca , W.C. Soares , and S. Chavez-Cerda , “Unveiling a truncated optical lattice associated with a triangular aperture using lights orbital angular momentum,” Phys Rev Lett. 105(5), 053904 (2010).
V.V. Kotlyar , S.N. Khonina , and V.A. Soifer , “Light field decomposition in angular harmonics by means of diffractive optics,” J Mod Opt. 45(7), 1495–1506 (1998).
A.V. Volyar , M.V. Brezko , Ya.E. Akimova , and Yu.A. Egorov , “Beyond the intensity or intensity moments and measuring the spectrum of optical vortices in complex beams,” Comput Opt. 42(5), 736–743 (2018).
V.V. Kotlyar , A.A. Kovalev , R.V. Skidanov , and V.A. Soifer , “Asymmetric Bessel-Gauss beams,” J. Opt. Soc. Am. A 31(9), 1977–1983 (2014).
V.V. Kotlyar , A.A. Kovalev , and E.G. Abramochkin , “Kummer laser beams with a transverse complex shift,” J Opt. 22(1), 015606 (2020). https://doi.org/10.1088/2040-8986/ab5ef1.
V.V. Kotlyar , and A.A. Kovalev , “Hermite-Gaussian modal laser beams with orbital angular momentum,” J. Opt. Soc. Am. A 31(2), 274–282 (2014).
V.V. Kotlyar , A.A. Kovalev , and A.P. Porfirev , “Vortex Hermite-Gaussian laser beams,” Opt Lett. 40(5), 701–704 (2015).
A.Y. Bekshaev , M.S. Soskin , and M.V. Vasnetsov , “Optical vortex symmetry breakdown and decomposition of the orbital angular momentum of light beams,” J. Opt. Soc. Am. A 20(8), 1635–1643 (2003).
V.G. Volostnikov , and E.G. Abramochkin , The modern optics of the Gaussian beams [In Russian] (Fizmatlit Publisher, Moscow, 2010).
V.V. Kotlyar , A.A. Kovalev , “Topological charge of asymmetric optical vortices,” Opt. Express 28(14), 20449–20460 (2020).
H. Wang , L. Liu , C. Zhou , J. Xu , M. Zhang , S. Teng , and Y. Cai , “Vortex beam generation with variable topological charge based on a spiral slit,” Nanophotonics 8(2), 317–324 (2019).
J. Leach , E. Yao , and M.J. Padgett , “Observation of the vortex structure of a non-integer vortex beam,” New J Phys. 6, 71 (2004).
A.J. Jesus-Silva , E.J.S. Fonseca , and J.M. Hickmann , “Study of the birth of a vortex at Fraunhofer zone,” Opt Lett. 37(21), 4552–4554 (2012).
J. Wen , L. Wang , X. Yang , J. Zhang , and S. Zhu , “Vortex strength and beam propagation factor of fractional vortex beams,” Opt. Express 27(4), 5893–5904 (2019).
A. Mourka , J. Baumgartl , C. Shanor , K. Dholakia , and E.M. Wright , “Visualization of the birth of an optical vortex using diffraction from a triangular aperture,” Opt. Express 19(7), 5760–5771 (2011).

G. Gibson , J. Courtial , M.J. Padgett , M. Vasnetsov , V. Pasko , S.M. Barnett , and S. Franke-Arnold , “Free-space information transfer using light beams carrying orbital angular momentum,” Opt. Express 12(22), 5448 (2004).
V.V. Kotlyar , A.A. Almazov , S.N. Khonina , V.A. Soifer , H. Elfstrom , and J. Turunen , “Generation of phase singularity through diffracting a plane or Gaussian beam by a spiral phase plate,” J. Opt. Soc. Am. A 22(5), 849–861 (2005).
“Handbook of mathematical functions,” Ed. by M. Abramowitz , and I.A. Stegun (National bureau of standards, Applied math, series, 55, Issue June (1964).
G.R. Salla , C. Perumangattu , S. Prabhakar , A. Anwar , and R.P. Singh , “Recovering the vorticity of a light beam after scattering,” Appl Phys Lett. 107(2), 021104 (2015).
E.G. Abramochkin , and V.G. Volostnikov , “Beam transformations and nontransformed beams,” Opt Commun. 83(1–2), 123–135 (1991).
V.V. Kotlyar , A.A. Kovalev , and V.A. Soifer , “Optical beams: Vortices with fractional orbital angular momentum,” in Chapter 8 in book “compendium of electromagnetic analysis” (World Scientific, NJ, 2020). 5, pp. 279–340. https://doi.org/10.1142/9789813270343.0008. ISBN 978-981-3270-16-9.
V.V. Kotlyar , A.A. Kovalev , A.G. Nalimov , and A.P. Porfirev , “Evolution of an optical vortex with an initial fractional topological charge,” Phys Rev A. 102, 023516 (2020).
A.V. Volyar , M.V. Brecko , Y.E. Akimova , and Y.A. Egorov , “Shaping and processing the vortex spectra of singular beams with anomalous orbital angular momentum,” Comput Opt. 43(4), 517–527 (2019).
A.G. White , C.P. Smith , N.R. Heckenberg , H. Rubinsztein-Dunlop , R. McDuff , C.O. Weiss , and C. Tamm , “Interferometric measurements of phase singularities in the output of a visible laser,” J Mod Opt. 38(12), 2531–2541 (1991).
A.G. Nalimov , V.V. Kotlyar , and V.A. Soifer , “Modeling of an image forming by a zone plate in X-ray,” Comput Opt. 35(3), 290–296 (2011).
G.E. Volovik , and V.P. Mineev , “Line and point singularitie in superfluid He3,” Pis'ma Zh Eksp Teor Fiz. 24, 605–608 (1976).
J. Jang , D.G. Ferguson , V. Vakarynk , R. Budakian , S.B. Chung , P.M. Goldbart , and Y. Maeno , “Observation of half-height magnetization spets in S2RuO4 ,” Science 331(6014), 186–188 (2011).
Y.G. Rubo , “Half vortices in exciton polariton condensates,” Phys Rev Lett. 99(10), 106401 (2007).
H. Flayac , I.A. Shelukh , D.D. Solnyshkov , and G. Malpuech , “Topological stability of the half-vortices in spinor exciton-polariton condensates,” Phys. Rev. B 81(4), 045318 (2010).
M. Krenn , R. Fickler , M. Fink , J. Handsteiner , M. Malik , T. Scheidl , R. Ursin , and A. Zeilinger , “Communication with spatially modulated light through turbulent air across Vienna,” New J Phys. 16(11), 113028 (2014).
J. Durnin , “Exact solutions for nondiffracting beams. I. The scalar theory,” J. Opt. Soc. Am. A 4(4), 651–654 (1987).
V.S. Vasilyev , A.I. Kapustin , R.V. Skidanov , V.V. Podlipnov , N.A. Ivliev , and S.V. Ganchevskaya , “Experimental investigation of the stability of Bessel beams in the atmosphere,” Comput Opt. 43(3), 376–384 (2019).
F. Wang , Y. Cai , H.T. Eyyuboglu , and Y. Baykal , “Average intensity and spreading of partially coherent standard and elegant Laguerre-Gaussian beams in turbulent atmosphere,” Prog Electromagn Res. 103, 33–55 (2010).
Y. Chen , F. Wang , C. Zhao , and Y. Cai , “Experimental demonstration of a Laguerre-Gaussian correlated Schell-model vortex beam,” Opt. Express 22(5), 5826–5838 (2014).
V.P. Lukin , P.A. Konyaev , and V.A. Sennikov , “Beam spreading of vortex beams propagating in turbulent atmosphere,” Appl Opt. 51(10), C84–C87 (2012).
K. Zhu , G. Zhou , X. Li , X. Zheng , and H. Tang , “Propagation of Bessel-Gaussian beams with optical vortices in turbulent atmosphere,” Opt. Express 16(26), 21315–21320 (2008).
S. Avramov-Zamurovic , C. Nelson , S. Guth , O. Korotkova , and R. Malek-Madani , “Experimental study of electromagnetic Bessel-Gaussian Schell model beams propagating in a turbulent channel,” Opt Commun. 359, 207–215 (2016).
I.P. Lukin , “Integral momenta of vortex Bessel-Gaussian beams in turbulent atmosphere,” Appl Opt. 55(12), B61–B66 (2016).

L.G. Wang , and W.W. Zheng , “The effect of atmospheric turbulence on the propagation properties of optical vortices formed by using coherent laser beam arrays,” J Opt A: Pure Appl Opt. 11(6), 065703 (2009).
M.V. Berry , “A note on superoscillations associated with Bessel beams,” J Opt. 15(4), 044006 (2013).
E. Ostrovsky , K. Cohen , S. Tsesses , B. Gjona , and G. Bartal , “Nanoscale control over optical singularities,” Optica 5(3), 283–288 (2018).
J.F. Nye , J.V. Hajnal , and J.H. Hannay , “Phase saddles and dislocations in two-dimensional waves such as the tides,” Proc. R. Soc. Lond. A 417(1852), 7–20 (1988).
J.F. Nye , Natural focusing and fine structure of light (Institute of Physics Publishing, Bristol, 1999).
M.V. Berry , “Wave dislocation reactions in non-paraxial Gaussian beams,” J Mod Opt. 45(9), 1845–1858 (1998).
F. Flossmann , U.T. Schwarz , and M. Maier , “Optical vortices in a Laguerre-Gaussian LG (1,0) beam,” J Mod Opt. 52(7), 1009–1017 (2005).
I.V. Basistiy , V.Yu. Bazhenov , M.S. Soskin , and M.V. Vasnetsov , “Optics of light beams with screw dislocations,” Opt Commun. 103(5–6), 422–428 (1993).
A.Ya. Bekshaev , M.S. Soskin , and M.V. Vasnetsov , “Centrifugal transformation of the transverse structure of freely propagating paraxial light beams,” Opt Lett. 31(6), 694–696 (2006).
A.Ya. Bekshaev , and A.I. Karamoch , “Spatial characteristics of vortex light beams produced by diffraction gratings with embedded phase singularity,” Opt Commun. 28(6), 1366–1374 (2008).
D. Rozas , C.T. Law , and G.A. Shwartzlander Jr , “Propagation dynamics of optical vortices,” J. Opt. Soc. Am. B 14(11), 3054–3065 (1997).
M. Abramowitz , and I.A. Stegun , Handbook of mathematical functions: With formulas, graphs, and mathematical tables (Dover Publications Inc, New York, 1979).
A. Longman , and R. Fedosejevs , “Optimal Laguerre-Gaussian modes for high-intensity optical vortices,” J. Opt. Soc. Am. A 37(5), 841–848 (2020).
V.V. Kotlyar , A.A. Kovalev , and A.P. Porfirev , “Topological stability of optical vortices diffracted by a random phase screen,” Comput Opt. 43(4), 917–925 (2019).
A.A. Kovalev , V.V. Kotlyar , and A.P. Porfirev , “Orbital angular momentum and topological charge of a multi-vortex Gaussian beam,” J. Opt. Soc. Am. A 37(11), 1740–1747 (2020).
G. Ruffato , “OAM-inspired new optics: The angular metalens,” Light Sci Appl. 10(1), 96 (2021).
Y. Guo , S. Zhang , M. Pu , Q. He , J. Jin , M. Xu , Y. Zhang , P. Gao , and X. Luo , “Spin-decoupled metasurface for simultaneous detection of spin and orbital angular momenta via momentum transformation,” Light Sci Appl. 10(1), 63 (2021).
Z. Jin , D. Janoschka , J. Deng , L. Ge , P. Dreher , B. Frank , G. Hu , J. Ni , Y. Yang , J. Li , G. Yu et al, “Phyllotaxis-inspired nanosieves with multiplexed orbital angular momentum,” eLight 1, 1 (2021).
D. Wei , Y. Cheng , R. Ni , Y. Zhang , X. Hu , S. Zhu , and M. Xiao , “Generating controllable Laguerre-Gaussian laser modes through intracavity spin-orbital angular momentum conversion of light,” Phys Rev Appl. 11(1), 014038 (2019).
V. Stella , T. Grogjeon , N. De Leo , L. Boarino , P. Munzerd , J.R. Lakowicz , and E. Descrovi , “Vortex beam generation by spin-orbit interaction with Bloch surface waves,” ACS Photonics 7(3), 774–783 (2020).
T. Arikawa , T. Hiraoka , S. Morimoto , F. Blanchard et al, “Transfer of optical angular momentum of light to plasmoni excitations in metamaterials,” Sci Adv. 6(24), 253 (2020).
V.V. Kotlyar , S.S. Stafeev , A.G. Nalimov , L. O'Faolain , and M.V. Kotlyar , “A dual-functionality metalens to shape a circularly polarized optical vortex or a second-order cylindrical vector beam,” Phot Nanostr Fund Appl. 43, 100898 (2021).
L. Zhu , M. Tang , H. Li , Y. Tai , and X. Li , “Optical vortex lattice: An explotation of orbital angular momentum,” Nanophotonics 10(9), 0139 (2021).
S. Fu , Y. Zhai , J. Zhang , X. Liu , R. Song , H. Zhou , and C. Gao , “Universal orbital angular momentum spectrum analyser for beams,” PhotoniX 1(1), 19 (2020).
D.M. Fatkhiev , M.A. Butt , E.P. Grakhova , R.V. Katluyarov , I.V. Stepanov , N.L. Kazanskiy , S.N. Khonina , V.S. Luubopytov , and A.K. Sultanov , “Recent advances in generation and detection of orbital angular momentum optical vortices: A review,” Sensors 21(15), 4988 (2021).

L. Zhu , and J. Wang , “A review of multiple optical vortices generation: Methods and applications,” Front Optoelectron. 12(1), 52–68 (2019).
D.L. Andrews , “Symmetry and quantum features in optical vortices,” Symmetry 13(8), 1368 (2021).
X. Wang , Z. Nie , Y. Liang , J. Wang , T. Li , and B. Jia , “Recent advances on optical vortex generation,” Nanophotonics 7, 1533–1556 (2018).
R. Chen , H. Zhou , M. Moretti , X. Wang , and J. Li , “Orbital angular momentum waves: Generation, detection and emerging applications,” IEEE Commun Surv Tutor. 22(2), 840–868 (2020).
A. Forbes , M. de Oliveira , and M.R. Dennis , “Structured light,” Nat Photon. 15(4), 253–262 (2021).
Y. Shen , X. Wang , Z. Xie , C. Min , X. Fu , Q. Liu , M. Gong , and X. Yuan , “Optical vortices 30 years on: OAM manipulation from topological charge to multiple singularities,” Light Sci Appl. 8, 90 (2019).
V.V. Kotlyar , and A.A. Kovalev , Accelerating and vortex laser beams. (CRC Press, Boca Raton, 2019), p. 298. https://doi.org/10.1201/9780429321610. ISBN: 9780429321610.
Frontires in optics and photonics, ed. by F. Capasso , and D. Couwenberg (de Gruyter GmbH, Berlin, 2021), p. 783.
G. Gbur , “Fractional vortex Hilbert's hotel,” Optica 3(3), 222–225 (2016).
I.V. Basistiy , M.S. Soskin , and M.V. Vasnetsov , “Optical wavefront dislocations and their properties,” Opt Commun. 119(5–6), 604–612 (1995).
B.Ya. Zel'dovich , N.D. Kundikova , F.V. Podgornov , and L.F. Rogacheva , “Formation of a ‘light flash' moving at an arbitrary velocity in vacuum,” Quantum Electron. 26(12), 1097–1099 (1996).
P. Saari , and I. Besieris , “Backward energy flow in simple 4-wave electromagnetic fields,” Eur J Phys. 42(5), 055301 (2021).
V.V. Kotlyar , S.N. Khonina , R.V. Skidanov , and V.A. Soifer , “Rotation of laser beams with zero of the orbital angular momentum,” Opt Commun. 274(1), 8 (2007).
A.V. Volyar , M.V. Bretsko , Ya.E. Akimova , and Yu.A. Egorov , “Avalanche instability of the orbital angular momentum higher order optical vortices,” Comput Opt. 43(1), 14 (2019).
V.V. Kotlyar , A.A. Kovalev , A.P. Porfirev , and E.S. Kozlova , “Orbital angular momentum of laser beam behied an off-axis spiral phase plate,” Opt Lett. 44(15), 3673 (2019).
G. Ruffalo , M. Massari , and F. Romanato , “Multiplication and division of the orbital angular momentum of light with diffractive transformation optics,” Light Sci Appl. 8, 113 (2019).
L. Gong , Q. Zhao , H. Zheng , X. Hu , K. Huang , J. Yang , and Y. Li , “Optical orbital-angular-momentum multiplexed data transmission under high scattering,” Light Sci Appl. 8, 27 (2019).
S.H. Kazemi , and M. Mahmoudi , “Identifying orbital angular momentum of light in quantum wells,” Laser Phys Lett. 16(7), 076001 (2019).
A.M. Konzelmann , S.O. Kruger , and H. Giessen , “Interaction of orbital angular momentum light with Rydberg excitons: Modifying dipole selection rules,” Phys. Rev. B 100(11), 115308 (2019).
F.N. Rybakov , and N.S. Kiselev , “Chiral magnetic skyrmions with arbitrary topological charge,” Phys. Rev. B 99(6), 064437 (2019).
L. Zhang , L. Zhang , and X. Liu , “Dynamical detection of topological charges,” Phys. Rev. A 99(5), 053606 (2019).
V.V. Kotlyar , and A.A. Kovalev , “Hermite–Gaussian modal laser beams with orbital angular momentum,” J. Opt. Soc. Am. A 31(2), 274 (2014).
S. Franke-Arnold , J. Leach , M.J. Padgett , V.E. Lembessis , D. Ellinas , A.J. Wright , J.M. Girkin , P. Ohberg , and A.S. Arnold , “Optical Ferris wheel for ultracold atoms,” Opt. Express 15(14), 8619 (2007).
T. Ando , N. Matsumoto , Y. Ohtake , Y. Takiguchi , and T. Inoue , “Structure of optical singularities in coaxial superpositions of Laguerre-Gaussian modes,” J. Opt. Soc. Am. A 27(12), 2602 (2010).
E.J. Galvez , N. Smiley , and N. Fernandes , “Composite optical vortices formed by collinear Laguerre-Gauss beams,” Proc. SPIE 6131, 613105 (2006).
C. Schulze , F.S. Roux , A. Dudley , R. Rop , M. Duparre , and A. Forbes , “Accelerated rotation with orbital angular momentum modes,” Phys. Rev. A 91(4), 043821 (2015).

L. Yang , D. Qian , C. Xin , Z. Hu , S. Ji , D. Wu , Y. Hu , J. Li , W. Huang , and J. Chu , “Two-photon polymerization of microstructures by a non-diffraction multifoci pattern generated from a superposed Bessel beam,” Opt Lett. 42(4), 743 (2017).

S. Supp , and J. Jahns , “Coaxial superposition of Bessel beams by discretized spiral axicons,” J Eur Opt Soc Rapid Publ. 14(1), 18 (2018).

S. Orlov , and A. Stabinis , “Propagation of superpositions of coaxial optical Bessel beams carrying vortices,” J Opt A: Pure Appl Opt. 6(5), S259 (2004).

M.V. Berry , “Quantal phase factors accompanying adiabatic changes,” Proc. R. Soc. Lond. A 392(1802), 45 (1984).

A.P. Prudnikov , Y.A. Brychkov , and O.I. Marichev , Integrals and series, special functions (Gordon and Breach, New York, 1981).

J.W. Goodman , Introduction to fourier optics, 2nd ed. (McGraw-Hill, New York, 1996).

V.V. Kotlyar , A.A. Kovalev , and A.G. Nalimov , “Conservation of the half-integer topological charge on propagation of a superposition of two Bessel-Gaussian beams,” Phys Rev A. 104(3), 033507 (2021).

A.J. Jesus-Silva , E.J.S. Fonseca , and J.M. Hickman , “Study of the birth of a vortex at Frauhofer zone,” Opt Lett. 37(12), 4552–4554 (2012).

J. Zeng , H. Zhang , Z. Xu , C. Zhao , Y. Cai , and G. Gbur , “Anomalous multi-ramp fractional vortex beams with arbitrary topological charge jumps,” Appl Phys Lett. 117(24), 241103 (2020).

A.A. Kovalev , and V.V. Kotlyar , “Optical vortex beams with the infinite topological charge,” J Opt. 23(5), 055601 (2021).

A.A. Kovalev , and V.V. Kotlyar , “Propagation-invariant laser beams with an array of phase singularities,” Phys. Rev. A 103(6), 063502 (2021).

E.G. Abramochkin , and V.G. Volostnikov , “Spiral-type beams: Optical and quantum aspects,” Opt Commun. 125(4–6), 302–323 (1996).

S. Rasouli , P. Amiri , V.V. Kotlyar , and A.A. Kovalev , “Characterization of a pair of superposed vortex beams having different winding numbers via diffraction from a quadratic curved-line grating,” J. Opt. Soc. Am. B 38(8), 2267–2276 (2021).

A.A. Kovalev , and V.V. Kotlyar , “Orbital angular momentum of superposition of identical shifted vortex beams,” J. Opt. Soc. Am. A 32(10), 1805–1810 (2015).

V.V. Kotlyar , A.A. Kovalev , P. Amiri , P. Soltani , and S. Rasouli , “Topological charge of two parallel Laguerre-Gaussian beams,” Opt. Express 29(26), 42962–42977 (2021).

V.V. Kotlyar , and A.A. Kovalev , Topological charge of optical vortices (Novaya Tekhnika, Samara, 2021), 185 pages, ISBN 978-5-88940-157-5.

O.O. Arkhelyuk , P.V. Polyanskii , A.A. Ivanovskii , and M.S. Soskin , “Creation and diagnostics of stable rainbow optical vortices,” Opt Appl. XXXIV(3), 419–426 (2004).

V. Denisenko , V. Shvedov , A.S. Desyatnikov , D.N. Neshev , W. Krolikovski , A. Volyar , M. Soskin , and Y.S. Kivshar , “Determination of topological charges of polychromatic optical vortices,” Opt. Express 17(26), 23374–23379 (2009).

D. Hakobyan , H. Magallanes , G. Seniutinas , S. Juodkazis , and E. Brasselet , “Tailoring orbital angular momentum of light in the visible domain with metallic metasurfaces,” Adv Opt Mater. 4(2), 306–312 (2015).

J. Kobashi , H. Yoshida , and M. Ozaki , “Polychromatic optical vortex generation from patterned cholesteric liquid crystals,” Phys Rev Lett. 116(25), 253903 (2016).

Y. Zhang , H. Guo , X. Qiu , X. Lu , X. Ren , and L. Chen , “LED-based chromatic and white-light vortices of fractional topological charges,” Opt Commun. 485, 126732 (2021).

G. Vallone , “On the properties of circular beams: Normalization, Laguerre-Gauss expansion, and free-space divergence,” Opt Lett. 40(8), 1717–1720 (2015).

G. Vallone , “Role of beam waist in Laguerre-Gauss expansion of vortex beams,” Opt Lett. 42(6), 1097–1100 (2017).

A.P. Prudnikov , Yu.A. Brychkov , and O.I. Marichev , Integrals and series, special functions (Taylor&Francis, London, 1992), p. 444.

G. Campbell , B. Hage , B. Buchler , and P. Lam , “Generation of high-order optical vortices using directly machined spiral phase mirrors,” Appl Opt. 51(7), 873–876 (2012).

Y. Chen , Z. Fang , Y. Ren , L. Gong , and R. Lu , “Generation and characterization of a perfect vortex beam with a large topological charge through a digital micromirror device,” Appl Opt. 54(27), 8030–8035 (2015).

C. Wang , Y. Ren , T. Liu , C. Luo , S. Qiu , Z. Li , and H. Wu , “Generation and measurement of high-order optical vortices by using the cross phase,” Appl Opt. 59(13), 4040–4047 (2020).
D. Chen , Y. Miao , H. Fu , H. He , J. Tong , and J. Dong , “High-order cylindrical vector beams with tunable topological charge up to 14 directly generated from a microchip laser with high beam quality and high efficiency,” APL Photonics 4(10), 106106 (2019). https://doi.org/10.1063/1.5119789.
P. Kumar , and N. Nishchal , “Modified Mach-Zehnder interferometer for determining the high-order topological charge of Laguerre-Gaussian vortex beams,” J. Opt. Soc. Am. A 36(8), 1447–1455 (2019).
Y. Li , Y. Han , and Z. Cui , “Measuring the topological charge of vortex beams with gradually changing-period spiral spoke grating,” IEEE Photonics Technol Lett. 32(2), 101–104 (2020).
I. Nape , B. Sephton , Y.-W. Huang , A. Vallés , C.-W. Qiu , A. Ambrosio , F. Capasso , and A. Forbes , “Enhancing the modal purity of orbital angular momentum photons,” APL Photon 5, 070802 (2020).
S. Hong , Y.S. Lee , H. Choi , C. Quan , Y. Li , S. Kim , and K. Oh , “Hollow silica photonic crystal fiber guiding 101 orbital angular momentum modes without phase distortion in C+ L band,” J Lightw Technol. 38(5), 1010–1018 (2020).
R. Fickler , G. Campbell , B. Buchler , P.K. Lam , and A. Zeilinger , “Quantum entanglement of angular momentum states with quantum numbers up to 10010,” Proc Natl Acad Sci U S A. 113(48), 13642–13647 (2016).
S. Fu , Y. Zhai , H. Zhou , J. Zhang , T. Wang , C. Yin , and C. Gao , “Demonstration of free-space one-to-many multicasting link from orbital angular momentum encoding,” Opt Lett. 44(19), 4753–4756 (2019).
Z. Qiao , Z. Wan , G. Xie , J. Wang , L. Qian , and D. Fan , “Multi-vortex laser enabling spatial and temporal encoding,” PhotoniX 1, 13 (2020).
J. Serna , and J. Movilla , “Orbital angular momentum of partially coherent beams,” Opt Lett. 26(7), 405–407 (2001).
E.G. Abramochkin , and V.G. Volostnikov , Modern optics of Gaussian beams (“Fizmatlit” Publisher, Moscow, 2010, in Russian).
V.V. Kotlyar , A.A. Kovalev , and A.P. Porfirev , “Vortex astigmatic Fourier-invariant Gaussian beams,” Opt. Express 27(2), 657–666 (2019).
P. Vaity , A. Aadhi , and R. Singh , “Formation of optical vortices through superposition of two Gaussian beams,” Appl Opt. 52(27), 6652–6656 (2013).
M.P. Backlund , M.D. Lew , A.S. Backer , S.J. Sahl , G. Grover , A. Agrawal , R. Piestun , and W.E. Moerner , “The double-helix point spread function enables precise and accurate measurement of 3D single-molecule localization and orientation,” Proc SPIE Int Soc Opt Eng. 8590, 85900L (2013).
J.M. Hickmann , E.J.S. Fonseca , W.C. Soares , and S. Chávez-Cerda , “Unveiling a truncated optical lattice associated with triangular aperture using light's orbital angular momentum,” Phys Rev Lett. 105(5), 053904 (2010).
L.E.E. de Araujo , and M.E. Anderson , “Measuring vortex charge with a triangular aperture,” Opt Lett. 36(6), 787–789 (2011).
P. Vaity , J. Banerji , and R.P. Singh , “Measuring the topological charge of an optical vortex by using a titled convex lens,” Phys. Lett. A 377(15), 1154–1156 (2013).
D. Shen , and D. Zhao , “Measuring the topological charge of optical vortices with a twisting phase,” Opt Lett. 44(9), 2334–2337 (2019).
G. Liu , K. Wang , Y. Lee , D. Wang , P. Li , F. Gou , Y. Li , C. Tu , S. Wu , and H. Wang , “Measurement of the topological charge and index of vortex vector optical fields with a space-variant half-wave plate,” Opt Lett. 43(4), 823–826 (2018).
B. Lan , C. Liu , D. Rui , M. Chen , F. Shen , and H. Xian , “The topological charge measurement of the vortex beam based on dislocation self-reference interferometry,” Phys Scr. 94(5), 055502 (2019).
B. Kodatskii , A. Sevryugin , E. Shalimov , I. Tursunov , and V. Venediktov , “Comparative study of reference wave lacking measurement of topological charge of the incoming optical vortex,” Proc. SPIE 11153, 111530G (2019).
S. Fu , Y. Zhai , J. Zhang , X. Liu , R. Song , H. Zhou , and C. Gao , “Universal orbital angular momentum spectrum analyzer for beams,” PhotoniX (Springer) 1, 19 (2020).

Z. Zhang , X. Qiao , B. Midya , K. Liu , J. Sun , T. Wu , W. Liu , R. Agarwal , J.M. Jornet , S. Longhi , N.M. Litchinitser , and L. Feng , “Tunable topological charge vortex microlaser,” Science 368(6492), 760–763 (2020).

K. Zhang , Y. Wang , Y. Yuan , and S.N. Burokur , “A review of orbital angular momentum vortex beams generation: from traditional methods to metasurfaces,” Appl Sci. 10, 1015 (2020).

K. Ciesielski , Set theory for the working mathematician (Cambridge University Press, Cambridge, 1997), https://doi.org/10.1017/CBO9781139173131.

A.A. Kovalev , and V.V. Kotlyar , “Orbital angular momentum of generalized cosine Gaussian beams with an infinite number of screw dislocations,” Optik 242, 166863 (2021).

S.N. Khonina , V.V. Kotlyar , V.A. Soifer , P. Paakkonen , and J. Turunen , “Measuring the light field orbital angular momentum using DOE,” Opt Mem Neural Netw. 10, 241–255 (2001).

T. Kaiser , D. Flamm , S. Schroter , and M. Duppare , “Complete modal decomposition of optical fibers using CGH-based correlation filter,” Opt. Express 17(11), 9347–9356 (2009).

M.V. Vasnetsov , J.P. Torres , D.V. Petrov , and L. Torner , “Observation of the orbital angular momentum spectrum of a light beam,” Opt Lett. 28(23), 2285–2287 (2003).

A. Forbes , A. Dudley , and M. McLaren , “Creation and detection of optical modes with spatial light modulators,” Adv Opt Phot. 8(2), 200 (2016).

A. D'Errico , R. D'Amelio , B. Piccirillo , F. Cardano , and L. Marrucci , “Measuring the complex orbital angular momentum spectrum and spatial mode decomposition of structured light beams,” Optica 4(11), 1350–1357 (2017).

J. Pinnell , V. Rodriguez-Fajardo , and A. Forbes , “Single-step shaping of the orbital angular momentum spectrum of light,” Opt. Express 27(20), 28009–28021 (2019).

Y. Yang , Q. Zhao , L. Liu , Y. Liu , C. Rosales-Guzman , and C. Qiu , “Manipulation of orbital angular momentum spectrum using pinhole plates,” Phys Rev Appl. 12(6), 064007 (2019).

J. Leach , M.J. Padgett , S.M. Barnett , S. Franke-Arnold , and J. Courtial , “Measuring the orbital angular momentum of a single photon,” Phys Rev Lett. 88(25 Pt 1), 257901 (2002).

A. Schulze , A. Dadley , D. Flamm , M. Duparre , and A. Forbes , “Measurements of the orbital angular momentum density of light by modal decomposition,” New J Phys. 15(7), 073025 (2013).

P. Bierdz , M. Kwon , C. Roncaioli , and H. Deng , “High fidelity detection of the orbital angular momentum of light by time mapping,” New J Phys. 15(11), 113062 (2013).

S. Li , P. Zhao , X. Feng , K. Cui , F. Liu , W. Zhang , and Y. Huang , “Measuring the orbital angular momentum spectrum with a single point detector,” Opt Lett. 43(19), 4607–4610 (2018).

E. Karimi , B. Piccirillo , E. Nagali , L. Marucci , and E. Santamato , “Efficient generation and sorting of orbital angular momentum eigenmodes of light by thermally tuned q-plates,” Appl Phys Lett. 94(23), 231124 (2009).

G.C.G. Berghout , M.P.J. Lavery , J. Cortial , M.W. Beijersbergen , and M.J. Padgett , “Efficient sorting of orbital angular momenyum states of light,” Phys Rev Lett. 105(15), 153601 (2010).

M. Mirhosseini , M. Malik , Z. Shi , and R.W. Boyd , “Efficient separation of the orbital angular momentum eigenstates of light,” Nat Commun. 4, 1–6 (2013).

H. Zucker , “Optical resonators with variable reflectivity mirrors,” Bell Syst Tech J. 49(9), 2349–2376 (1970).

E. Karimi , G. Zito , B. Piccirillo , L. Marrucci , and E. Santamato , “Hypergeometric-Gaussian modes,” Opt Lett. 32(21), 3053–3055 (2007).

V.V. Kotlyar , A.A. Kovalev , and A.P. Porfirev , “Elliptic Gaussian optical vortices,” Phys. Rev. A 95(5), 053805 (2017).

E.G. Abramochkin , and V.G. Volostnikov , “Beam transformation and nontransformed beams,” Opt Commun. 83(1–2), 123–135 (1991).

S.N. Khonina , V.V. Kotlyar , V.A. Soifer , P. Paakkonen , J. Simonen , and J. Turunen , “An analysis of the angular momentum of a light field in terms of angular harmonics,” J Mod Opt. 48(10), 1543–1557 (2001).

Y. Yang , L. Wu , Y. Liu , D. Xie , Z. Jin , J. Li , G. Hu , and C. Qiu , “Deuterogenic plasmonic vortices,” Nano Lett. 20(9), 6774–6779 (2020).

D.V. Petrov , “Vortex-edge dislocation interaction in a linear medium,” Opt Commun. 188(5–6), 307–312 (2001).

D.V. Petrov , “Splitting of an edge dislocation by an optical vortex,” Opt Quant Electr. 34(8), 759–773 (2002).

M.V. Berry , “Optical currents,” J Opt A: Pure Appl Opt. 11(9), 094001 (2009).
V.V. Kotlyar , and A.A. Kovalev , “Optical vortex beams with a symmetric and almost symmetric OAM spectrum,” J. Opt. Soc. Am. A 38(9), 1276–1283 (2021).
I. Hebri , and S. Rasouli , “Combined half-integer Bessel-like beams: A set of solutions of the wave equation,” Phys. Rev. A 98(4), 043826 (2018).
A. Vasara , J. Turunen , and A.T. Friberg , “Realization of general nondiffracting beams with computer-generated holograms,” J. Opt. Soc. Am. A 6(11), 1748–1754 (1989).
N.R. Heckenberg , R. McDaff , C.P. Smith , and A.G. White , “Generation of optical phase singularities by computer-generated holograms,” Opt Lett. 17(3), 221–223 (1992).
V.Yu. Bazhenov , M.S. Soskin , and M.V. Vasnetsov , “Screw dislocations in light wavefronts,” J Mod Opt. 39(5), 985–990 (1992).
V.V. Kotlyar , S.N. Khonina , G.V. Uspleniev , M.V. Shinkarev , and V.A. Soifer , “The phase rotor filter,” J Mod Opt. 39(5), 1147–1154 (1992).
M.V. Beijersbergen , R.P.C. Coerwinkel , M. Kristensen , and J.P. Woerdman , “Helical-wavefront laser beams produced with a spiral phase plate,” Opt Commun. 112(5–6), 321–327 (1994).
A. Fedotowsky , and K. Lehovec , “Optimal filter design for annular imaging,” Appl Opt. 13(12), 2919–2923 (1974).
S.N. Khonina , V.V. Kotlyar , V.A. Soifer , G.V. Uspleniev , and M.V. Shinkarev , “Trochoson,” Opt Commun. 91(3–4), 158–162 (1992).
L. Marrucci , C. Manzo , and D. Paparo , “Optical spin-to-orbital angular momentum conversion in inhomogeneous anisotropic media,” Phys Rev Lett. 96(16), 163905 (2006).
D.N. Naik , and N.K. Viswanahan , “Generation of singular optical beams from fundamental Gaussian beam using Sagnac interferometer,” J Opt. 18(9), 095601 (2016).
L. Zhu , and J. Wang , “Arbitrary manipulation of spatial amplitude and phase using phase-only spatial light modulators,” Sci Rep. 4, 7441 (2015).
R.C. Devlin , A. Ambrosio , A. Rubin , J.B. Mueller , and F. Capasso , “Arbitrary spin-to-orbital angular momentum conversion of light,” Science 358(6365), 896–901 (2017).
P. Miao , Z. Zhang , J. Sun , W. Walasik , S. Longhi , N.M. Litchintser , and L. Feng , “Orbital angular momentum microlaser,” Science 353(6298), 464–467 (2016).
J. Zhang , Z. Guo , R. Li , W. Wang , A. Zhang , J. Liu , S. Qu , and J. Gao , “Circular polarization analyzer based on the combined coaxial Archimedes' spiral structure,” Plasmonics 10(6), 1256–1261 (2015).
J. Zhang , Z. Guo , K. Zhou , L. Ran , L. Zhu , W. Wang , Y. Sun , F. Shen , J. Gao , and S. Liu , “Circular polarization analyzer based on an Archimedean nano-pinholes array,” Opt. Express 23(23), 30523–30531 (2015).
H. Zhan , J. Li , K. Guo , and Z. Guo , “Generation of acoustic vortex beams with designed Fermat's spiral diffraction grating,” J Acoust Soc Am. 146(6), 4237–4243 (2019).
D. He , H. Yan , and B. Lu , “Interaction of the vortex and edge dislocation embedded in a cosh-Gaussian beam,” Opt Commun. 282(20), 4035–4044 (2009).
H. Chen , W. Wang , Z. Gao , and W. Li , “Splitting of an edge dislocation by a vortex emergent from a nonparaxial beam,” J. Opt. Soc. Am. B 36(10), 2804–2809 (2019).
A.T. O'Neil , and J. Courtial , “Mode transformations in terms of the constituent Hermite-Gaussian or Laguerre-Gaussian modes and the variable-phase mode converter,” Opt Commun. 181(1–3), 35–45 (2000).
M.J. Padgett , and L. Allen , “Orbital angular momentum exchange in cylindrical-lens mode converters,” J. Opt. B: Quant. Semicl 4(2), S17–S19 (2002).
R. Zeng , and Y. Yang , “Generation of an asymmetric optical vortex array with tunable singularity distribution,” J. Opt. Soc. Am. A 38(3), 313–320 (2021).
H. Fan , H. Zhang , C. Cai , M. Tang , H. Li , J. Tang , and X. Li , “Flower-shaped optical vortex array,” Ann Phys. 533, 2000573 (2021).
J. Jin , X. Li , M. Pu , Y. Guo , P. Gao , M. Xu , Z. Zhang , and X. Luo , “Angular-multiplexed nultichannel optical vortex arrays generators based on geometric metasurface,” iScince 24, 102107 (2021).
V.V. Kotlyar , A.A. Kovalev , A.P. Porfirev , and E.S. Kozlova , “Three different types of astigmatic Hermite-Gaussian beams with orbital angular momentum,” J Opt. 21(11), 115601 (2019).

J. Lin , J. Dellinger , P. Genevet , B. Cluzel , F. de Fornel , and F. Capasso , “Cosine-Gauss plasmon beam: A localized long-range nondiffracting surface wave,” Phys Rev Lett. 109(9), 093904 (2012).
V. Kotlyar , and A. Kovalev , “Converting an array of edge dislocations into a multivortex beam,” J. Opt. Soc. Am. A 38(5), 719–726 (2021).
B. Lu , and P. Wu , “Analytical propagation equation of astigmatic Hermite-Gaussian beams through a 4x4 paraxial optical systems and their symmetrizing transformation,” Opt Las Technol. 35, 497–504 (2003).
Y.F. Chen , C.C. Chay , C.Y. Lee , J.C. Tung , H.C. Liang , and K.T. Huang , “Characterizing the propagation evolution of wave patterns and vortex structures in astigmatic transformations of Hermite-Gaussian beams,” Las Phys. 28, 015002 (2017).
E.G. Abramochkin , E.U. Razueva , and V.G. Volostnikov , “Hermite-Laguerre-Gaussian beams in astigmatic optical systems,” Proc. SPIE 7009, 70090M (2008).
A.Y. Bekshaev , M.S. Soskin , and M.V. Vasnetsov , “Transformation of higher-order optical vortices upon focusing by an astigmatic lens,” Opt Commun. 241(4–6), 237–247 (2004).
A.Y. Bekshaev , and A.I. Karamoch , “Astigmatic telescopic transformation of a high-order optical vortex,” Opt Commun. 281(23), 5687–5696 (2008).
K. Zhu , J. Zhu , Q. Su , and H. Tang , “Propagation properties of an astigmatic sin-Gaussian beam in strongly nonlocal nonlinear media,” Appl Sci. 9(1), 71 (2019).
T.D. Huang , and T.H. Lu , “Large astigmatic laser cavity modes and astigmatic compensation,” Appl. Phys. B 124(5), 72 (2018).
J. Pan , Y. Shen , Z. Wan , X. Fu , H. Zhang , and Q. Liu , “Index-tunable structured-light beams from a laser with a intracavity astigmatic mode converter,” Phys Rev Appl. 14(4), 044048 (2020).
H. Chen , Z. Gao , H. Yang , S. Xiao , F. Wang , X. Huang , and X. Liu , “Evolution behavior of two edge dislocations passing through an astigmatic lens,” J Mod Opt. 59(21), 1863–1872 (2012).
H. Yan , and B. Lu , “Vortex-edge dislocation interaction in the presence of an astigmatic lens,” Opt Commun. 282(5), 717–726 (2009).
V.V. Kotlyar , A.A. Kovalev , and A.G. Nalimov , “Converting an nth-order edge dislocation to a set of optical vortices,” Optik 243, 167453 (2021).
A.M. Sedletsky , “Asymptotics of nulls of a degenerate hypergeometric function,” Mathematicheskii Zametki 82(2), 262–271 (2007).
J.H. McLeod , “The axicon: A new type of optical element,” J Opt Soc Am. 44(8), 592–597 (1954).
G. Indebetouw , “Nondiffracting optical fields: Some remarks on their analysis and synthesis,” J. Opt. Soc. Am. A 6(1), 150–152 (1989).
V.V. Kotlyar , A.A. Kovalev , and A.P. Porfirev , “Optimal phase element for generating a perfect optical vortex,” J. Opt. Soc. Am. A 33(12), 2376–2384 (2016).
C.E.R. Caron , and R.M. Potuliege , “Bessel-modulated Gaussian beams with quadratic radial dependence,” Opt Commun. 164(1–3), 83–93 (1999).
G. Li , H. Lee , and E. Wolf , “New generalized Bessel-Gauss laser beams,” J. Opt. Soc. Am. A 21, 640–646 (2004).
V.V. Kotlyar , A.A. Kovalev , and V.A. Soifer , “Hankel-Bessel laser beams,” J. Opt. Soc. Am. A 29(5), 741–747 (2012).
Y. Ismail , N. Khilo , V. Belyi , and A. Forbes , “Shape invariant higher-order Bessel-like beams carrying orbital angular momentum,” J Opt. 14(8), 085703 (2012).
S.C. Pei , and J.J. Ding , “Eigenfunctions of linear canonical transform,” IEEE Trans Signal Process. 50(1), 11–26 (2002).
A.P. Prudnikov , Y.A. Brychkov , and O.I. Marichev , Integrals and series, volume 2, special functions (Gordon and Breach, New York, 1986).
V.V. Kotlyar , A.A. Kovalev , and D.S. Kalinkina , “Fractional-order-Bessel Fourier-invariant optical vortices,” Opt Commun. 492, 126974 (2021).
V.V. Kotlyar , and A.A. Kovalev , “Family of hypergeometric laser beams,” J. Opt. Soc. Amer. A 25(1), 262–270 (2008). https://doi.org/10.1364/JOSAA.25.000262.
E. Karimi , B. Piccirillo , L. Marrucci , and E. Santamato , “Improved focusing with hypergeometric-Gaussian type-II optical modes,” Opt. Express 16(25), 21069–21075 (2008). https://doi.org/10.1364/OE.16.021069.

V.V. Kotlyar , A.A. Kovalev , R.V. Skidanov , S.N. Khonina , and J. Turunen , “Generating hypergeometric laser beams with a diffractive optical element,” Appl Opt. 47(32), 6124–6133 (2008). https://doi.org/10.1364/AO.47.006124.
S.N. Khonina , S.A. Balalaev , R.V. Skidanov , V.V. Kotlyar , B. Paivanranta , and J. Turunen , “Encoded binary diffractive element to form hypergeometric laser beams,” JOpt A: Pure Appl Opt. 11(6), 065702 (2009). https://doi.org/10.1088/1464-4258/11/6/065702.
V.V. Kotlyar , and A.A. Kovalev , “Nonparaxial hypergeometric beams,” J Opt A: Pure Appl Opt. 11(4), 045711 (2009). https://doi.org/10.1088/1464-4258/11/4/045711.
V.V. Kotlyar , A.A. Kovalev , and V.A. Soifer , “Lensless focusing of hypergeometric laser beams,” J Opt. 13(7), 075703 (2011). https://doi.org/10.1088/2040-8978/13/7/075703.
B. de Lima Bernardo , and F. Moraes , “Data transmission by hypergeometric modes through a hyperbolic-index medium,” Opt. Express 19(12), 11264–11270 (2011). https://doi.org/10.1364/OE.19.011264.
V.V. Kotlyar , A.A. Kovalev , and A.G. Nalimov , “Propagation of hypergeometric laser beams in a medium with a parabolic refractive index,” J Opt. 15(12), 125706 (2013). https://doi.org/10.1088/2040-8978/15/12/125706.
L. Bian , and B. Tang , “Propagation properties of hypergeometric-Gaussian type-II beams through the quadratic-index medium,” Appl Opt. 57(17), 4735–4742 (2018). https://doi.org/10.1364/AO.57.004735.
B. Tang , C. Jiang , and H. Zhu , “Fractional Fourier transform for confluent hypergeometric beams,” Phys. Lett. A 376(38–39), 2627–2631 (2012). https://doi.org/10.1016/j.physleta.2012.07.017.
J. Peng , Z. Shan , Y. Yuan , Z. Cui , W. Huang , and J. Qu , “Focusing properties of hypergeometric Gaussian beam through a high numerical-aperture objective,” Prog Electromagn Res. 51, 21–26 (2015). https://doi.org/10.2528/pierl14101304.
T. Bin , J. Chun , Z. Haibin , Z. Xin , and W. Shuai , “The propagation of hypergeometric beams through an annular apertured paraxial ABCD optical system, Laser Phys. 24(12), 125002 (2014). https://doi.org/10.1088/1054-660x/24/12/125002.
J. Li , and Y. Chen , “Propagation of confluent hypergeometric beam through uniaxial crystals orthogonal to the optical axis,” Opt Laser Technol. 44(5), 1603–1610 (2012). https://doi.org/10.1016/j.optlastec.2011.11.041.
Y. Zhu , L. Zhang , Z. Hu , and Y. Zhang , “Effects of non-Kolmogorov turbulence on the spiral spectrum of hypergeometric-Gaussian laser beams,” Opt. Express 23(7), 9137–9146 (2015). https://doi.org/10.1364/OE.23.009137.
X. Wang , L. Wang , B. Zheng , Z. Yang , and S. Zhao , “Effects of oceanic turbulence on the propagation of hypergeometric-Gaussian beam carrying orbital angular momentum.” In Proceedings of the 2020 IEEE international conference on communications workshops, Dublin, Ireland, 7–11 June 2020. https://doi.org/10.1109/iccworkshops49005.2020.9145217.
X. Wang , L. Wang , and S. Zhao , “Research on hypergeometric-Gaussian vortex beam propagating under oceanic turbulence by theoretical derivation and numerical simulation,” J Mar Sci Eng. 9(4), 442 (2021). https://doi.org/10.3390/jmse9040442.
L. Bian , and B. Tang , “Evolution properties of hypergeometric-Gaussian type-II beams in strongly nonlocal nonlinear media,” J. Opt. Soc. Am. B 35(6), 1362–1367 (2018). https://doi.org/10.1364/JOSAB.35.001362.
T. Bin , B. Lirong , Z. Xin , and C. Kai , “Propagation of hypergeometric Gaussian beams in strongly nonlocal nonlinear media,” Laser Phys. 28(1), 015001 (2018). https://doi.org/10.1088/1555-6611/aa9628.
A.A. Kovalev , V.V. Kotlyar , and A.P. Porfirev , “Auto-focusing accelerating hyper-geometric laser beams,” J Opt. 18(2), 025610 (2016). https://doi.org/10.1088/2040-8978/18/2/025610.
Y. Zhu , Y. Zhang , and G. Yang , “Evolution of orbital angular momentum mode of the autofocusing hypergeometric-Gaussian beams through moderate-to-strong anisotropic non-Kolmogorov turbulence,” Opt Commun. 405, 66–72 (2017). https://doi.org/10.1016/j.optcom.2017.07.047.
A.A.A. Ebrahim , F. Saad , and L. Ez-zariy, “Theoretical conversion of the hypergeometric-Gaussian beams family into a high-order spiraling Bessel beams by a curved fork-shaped hologram,” Opt Quant Electron. 49, 169 (2017). https://doi.org/10.1007/s11082-017-0987-6.
G. Jin , L. Bian , L. Huang , and B. Tany , “Radiation forces of hypergeometric-Gaussian type-II beams acting on a Rayleigh dielectric sphere,” Opt Las Techn. 126, 106124 (2020). https://doi.org/10.1016/j.optlastec.2020.106124.

R.L. Phillips , and L.C. Andrews , “Spot size and divergence for Laguerre-Gaussian beams of any order,” Appl Opt. 22(5), 643–644 (1983). https://doi.org/10.1364/AO.22.000643.
C. Alpmann , C. Scholer , and C. Denz , “Elegant Gaussian beams for enhanced optical manipulation,” Appl Phys Lett. 106(24), 241102 (2015). https://doi.org/10.1063/1.4922743.
N. Nossir , L. Dalil-Essakali , and A. Belafhal , “Optical trapping of particles by radiation forces of doughnut laser beams in the Rayleigh regime,” Opt Quantum Electron. 53(2), 100 (2021). https://doi.org/10.1007/s11082-021-02752-y.
J. Su ,, N. Li , J. Mou , Y. Liu , X. Chen , and H. Hu , “Simultaneous trapping of two types of particles with focused elegant third-order hermite–Gaussian beams,” Micromachines 12(7), 769 (2021). https://doi.org/10.3390/mi12070769.
D. Wei , S. Li , J. Zeng , X. Zhu , T. Chen , Y. Cai , and J. Yu , “Comparative study of spiral spectrum of elegant and standard Laguerre–Gaussian beams in atmospheric turbulence,” J Russ Laser Res. 41(4), 364–372 (2020). https://doi.org/10.1007/s10946-020-09887-5.
M. Dong , and Y. Yang , “Coherent vortices properties of partially coherent elegant laguerre-Gaussian beams in the free space,” Opt. Photonics News 10(6), 159–166 (2020). https://doi.org/10.4236/opj.2020.106017.
E.G. Abramochkin , and V.G. Volostnikov , “Generalized Gaussian beams,” J Opt A: Pure Appl Opt. 6(5), S157–S161 (2004).
V.V. Kotlyar , A.A. Kovalev , and A.P. Porfirev , “Vortex Hermite–Gaussian laser beams,” Opt Lett. 40(5), 701–704 (2015).
M. Bock , S. Das , and R. Grunwald , “Ultrashort highly localized wavepackets,” Opt. Express 20(11), 12563–12578 (2012).
X. Liu , and J. Pu , “Investigation on the scintillation reduction of elliptical vortex beams propagating in atmospheric turbulence,” Opt. Express 19(27), 26444–26450 (2011).
L.C. Andrews , and R.L. Phillips , Laser beam propagation through random media (SPIE Press, Bellingham, Washington, 1998).
X. Zhang , T. Xia , S. Cheng , and S. Tao , “Free-space information transfer using the elliptic vortex beam with fractional topological charge,” Opt Commun. 431, 238–244 (2019).
K. Wu , Y. Huai , T. Zhao , and Y. Jin , “Propagation of partially coherent four-petal elliptic Gaussian vortex beams in atmospheric turbulence,” Opt. Express 26(23), 30061–30075 (2018).
R.V. Skidanov , and M.A. Rykov , “The modification of laser beam for optimization of optical trap force characteristics,” Comput Opt. 37(4), 431–435 (2013).
D.A. Belousov , A.V. Dostovalov , V.P. Korolkov , and S.L. Mikerin , “A microscope image processing method for analyzing TLIPSS structures,” Comput Opt. 43(6), 936–945 (2019).
A.V. Dostovalov , K.A. Okotrub , K.A. Bronnikov , V.S. Terentyev , V.P. Korolkov , and S.A. Babin , “Influence of femtosecond laser pulse repetition rate on thermochemical laser-induced periodic surface structures formation by focused astigmatic Gaussian beam,” Laser Phys Lett. 16(2), 026003 (2019).
A.V. Dostovalov , T.J.Y. Derrien , S.A. Lizunov , F. Přeučil , K.A. Okotrub , T. Mocek , V.P. Korolkov , S.A. Babin , and N.M. Bulgakova , “LIPSS on thin metallic films: New insights from multiplicity of laser-excited electromagnetic modes and efficiency of metal oxidation,” Appl Surf Sci. 491, 650–658 (2019).
Y. Cai , and Q. Lin , “Decentered elliptical Gaussian beam,” Appl Opt. 41(21), 4336–4340 (2002).
V.V. Kotlyar , A.A. Kovalev , and A.P. Porfirev , “Astigmatic laser beams with a large orbital angular momentum,” Opt. Express 26(1), 141–156 (2018).
P.S. Bullen , “The power means,” in Handbook of means and their inequalities (Kluwer, Dordrecht, 2003), pp. 175–265. https://doi.org/10.1007/978-94-017-0399-4.
M.V. Berry , M.R. Jeffrey , and M. Mansuripur , “Orbital and spin angular momentum in conical diffraction,” J Opt A: Pure Appl Opt. 2005, 7(11), 685–690 (2005).
Z. Mei , O. Korotkova , D. Zhao , and Y. Mao , “Self-focusing vortex beams,” Opt Lett. 46(10), 2384–2387 (2021).
S. Goorden , J. Bertolotti , and A. Mosk , “Superpixel-based spatial amplitude and phase modulation using a digital micromirror device,” Opt. Express 22(15), 17999–18009 (2014).
O. Mendoza-Yero , G. Mínguez-Vega , and J. Lancis , “Encoding complex fields by using a phase-only optical element,” Opt Lett. 39(7), 1740–1743 (2014).
A.A. Kovalev , V.V. Kotlyar , and D.S. Kalinkina , “Propagation-invariant off-axis elliptic Gaussian beams with the orbital angular momentum,” Photonics 8(6), 190 (2021).

Z. Liu , Y. Liu , Y. Ke , Y. Liu , W. Shu , H. Luo , and S. Wen , “Generation of arbitrary vector vortex beams on hybrid-order Poincare sphere,” Photon Res. 5(1), 15–21 (2017). https://doi.org/10.1364/PRJ.5.000015.
S. Fu , Y. Zhai , T. Wang , C. Yin , and C. Gao , “Tailoring arbitrary hybrid Poincare beams through a single hologram,” Appl Phys Lett. 111(21), 211101 (2017). https://doi.org/10.1063/1.5008954.
Y. Zhang , P. Chen , S. Ge , T. Wei , J. Tang , W. Hu , and Y. Lu , “Spin-controlled massive channels of hybrid-order Poincare sphere beams,” Appl Phys Lett. 117(8), 081101 (2020). https://doi.org/10.1063/5.0020398.
J. Liu , X. Chen , Y. He , L. Lu , H. Ye , G. Chai , S. Chen , and D. Fan , “Generation of arbitrary cylindrical vector vortex beams with cross-polarized modulation,” Res Phys. 19, 103455 (2020). https://doi.org/10.1016/j.rinp.2020.103455.
G. Arora , S. Deepa , S.N. Khan , and P. Senthilkumaran , “Detection of degenerate Stokes index states,” Sci Rep. 10(1), 20759 (2020). https://doi.org/10.1038/s41598-020-77365-8.
G. Arora , Ruchi, and P. Senthilkumaran , “Hybrid order poincare spheres for stokes singularities,” Opt Lett. 45(18), 5136–5139 (2020). https://doi.org/10.1364/OL.400946.
S.S. Stafeev , V.V. Kotlyar , A.G. Nalimov , M.V. Kotlyar , and L. O'Faolain , “Subwavelength gratings for polarization conversion and focusing of laser light,” Photonics Nanostructures: Fundam Appl. 27, 32–41 (2017). https://doi.org/10.1016/j.photonics.2017.09.001.
P. Lochab , P. Senthilkumaran , and K. Khare , “Designer vector beams maintaining a robust intensity profile on propagation through turbulence,” Phys. Rev. A 98(2), 023831 (2018). https://doi.org/10.1103/PhysRevA.98.023831.
M. Berry , “Geometry of phase and polarization singularities illustrated by edge diffraction and the fides,” Proc. SPIE 4403 (2001). https://doi.org/10.1117/12.428252.
I. Freund , “Polarization singularity indices in Gaussian laser beams,” Opt Commun. 201(4–6), 251–270 (2002). https://doi.org/10.1016/S0030-4018(01)01725-4.
A.A. Kovalev , and V.V. Kotlyar , “Gaussian beams with multiple polarization singularities,” Opt Commun. 423, 111–120 (2018). https://doi.org/10.1016/j.optcom.2018.04.023.
M. Born , and E. Wolf , Principles of optics (Pergamon Press, Oxford, 1968).
Q. Zhan , “Cylindrical vector beams: From mathematical concepts to applications,” Adv Opt Photon. 1(1), 1–57 (2009). https://doi.org/10.1364/AOP.1.000001.
V.V. Kotlyar , S.S. Stafeev , and A.A. Kovalev , “Sharp focusing of a light field with polarization and phase singularities of an arbitrary order,” Comput Opt. 43(3), 337–346 (2019). https://doi.org/10.18287/2412-6179-2019-43-3-337-346.
B. Richards , and E. Wolf , “Electromagnetic diffraction in optical systems. II. Structure of the image field in an aplanatic system,” Proc. R. Soc. Lond. A 253(1274), 358–379 (1959). https://doi.org/10.1098/rspa.1959.0200.
V.V. Kotlyar , A.A. Kovalev , S.S. Stafeev , A.G. Nalimov , and S. Rasouli , “Tightly focusing vector beams containing V-points polarization singularities,” Opt Las Techn. 145 107479 (2022).
F. Cardano , E. Karimi , S. Slussarenko , L. Marrucci , C. de Lisio , and E. Santamato , “Polarization pattern of vector vortex beams generated by q-plates with different topological charges,” Appl Opt. 51(10), C1–C6 (2012). https://doi.org/10.1364/AO.51.0000C1.
F. Cardano , E. Karimi , L. Marrucci , C. de Lisio , and E. Santamato , “Generation and dynamics of optical beams with polarization singularities,” Opt. Express 21(7), 8815–8820 (2013). https://doi.org/10.1364/OE.21.008815.
P. Kumar , S. Pal , N. Nishchal , and P. Senthilkumaran , “Non-interferometric technique to realize vector beams embedded with polarization singularities,” J. Opt. Soc. Am. A 37(6), 1043–1052 (2020). https://doi.org/10.1364/JOSAA.393027.
B. Khajavi , and E.J. Galvez , “High-order disclinations in space-variant polarization,” J Opt. 18(8), 084003 (2016). https://doi.org/10.1088/2040-8978/18/8/084003.
L. Lu , Z. Wang , and Y. Cai , “Propagation properties of phase-locked radially-polarized vector fields array in turbulent atmosphere,” Opt. Express 29(11), 16833–16844 (2021). https://doi.org/10.1364/OE.427003.
M. Meier , V. Romano , and T. Feurer , “Material processing with pulsed radially and azimuthally polarized laser radiation,” Appl. Phys. A 86(3), 329–334 (2007). https://doi.org/10.1007/s00339-006-3784-9.
P. Meng , S. Pereira , and P. Urbach , “Confocal microscopy with a radially polarized focused beam,” Opt. Express 26(23), 29600–29613 (2018). https://doi.org/10.1364/OE.26.029600.

L. Carretero , P. Acebal , and S. Blaya , “Three-dimensional analysis of optical forces generated by an active tractor beam using radial polarization,” Opt. Express 22(3), 3284–3295 (2014). https://doi.org/10.1364/OE.22.003284.

J.F. Nye , “Polarization effects in the diffraction of electromagnetic waves: The role of disclinations,” Proc R Soc Lond. 387, 105–132 (1983).

X. Wang , J. Ding , W. Ni , C. Guo , and H. Wang , “Generation of arbitrary vector beams with a spatial light modulator and a common path interferometric arrangement,” Opt Lett. 32(24), 3549–3551 (2007).

D. Naidoo , F.S. Roux , A. Dudley , I. Litvin , B. Piccirillo , and L. Marrucci , “Controlled generation of higher-order poincare sphere beams from a laser,” Nat Phot. 10(5), 327–333 (2016).

Z. Bomzon , G. Biener , V. Kleiner , and E. Hasman , “Radially and azimuthally polarized beams generated by space-variant dielectric subwavelength gratings,” Opt Lett. 27(5), 285–287 (2002).

V.V. Kotlyar , S.S. Stafeev , A.G. Nalimov , and L. O'Faolain , “Subwavelength grating-based metalens for focusing of laser light,” Appl Phys Lett. 114(14), 141107 (2019).

Z. Ren , Z. Chen , X. Wang , J. Ding , and H. Wang , “Polarization interferometric prism: A versatile tool for generation of vector fields, measurement of topological charges, and implementation of a spin-orbit controlled-not gate,” Appl Phys Lett. 118(1), 011105 (2021).

V. Kumar , and N.K. Viswanathan , “Topological structures in the poynting vector field: An experimental realization,” Opt Lett. 38(19), 3886–3889 (2013).

M. Born , and E. Wolf , “Principles of optics,” Pergamon Press, Oxford, p. 836 (1980).

G. Arora , Ruchi, P. Senthilkumaran , “Hybrid order poincare spheres for stoks singularities,” Opt Lett. 45(18), 5136 (2020).

V.V. Kotlyar , A.G. Nalimov , and S.S. Stafeev , “Inversion of the axial projection of the spin angular momentum in the region of the backward energy flow in sharp focus,” Opt. Express 28(23), 33830–33839 (2020).

T. Bauer , M. Neugebauer , G. Leuchs , and P. Banzer , “Optical polarization Möbius strips and points of purely transverse spin density,” Phys Rev Lett. 117(1), 013601 (2016).

T. Bauer , P. Banser , E. Karimi , S. Orlov , A. Rubano , L. Marrucci , E. Santamato , R.W. Boyd , and G. Leuchs , “Observation of optical polarization Möbius strips,” Science 347(6225), 964–966 (2015).

V.V. Kotlyar , A.G. Nalimov , A.A. Kovalev , A.P. Porfirev , and S.S. Stafeev , “Spin-orbit and orbit-spin conversion in the sharp focus of laser light: Theory and experiment,” Phys. Rev. A 102(3), 033502 (2020).

V.V. Kotlyar , S.S. Stafeev , and A.G. Nalimov , “Sharp focusing of a hybrid vector beam with a polarization singularity,” Photonics 8(6), 227 (2021).